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Summing the Instantons: Quantum Cohomology and Mirror Symmetry in Toric Varieties

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Abstract

We use the gauged linear sigma model introduced by Witten to calculate instanton expansions for correlation functions in topological sigma models with target space a toric variety V or a Calabi–Yau hypersurface $M \subset V$. In the linear model the instanton moduli spaces are relatively simple objects and the correlators are explicitly computable; moreover, the instantons can be summed, leading to explicit solutions for both kinds of models. In the case of smooth V, our results reproduce and clarify an algebraic solution of the V model due to Batyrev. In addition, we find an algebraic relation determining the solution for M in terms of that for V. Finally, we propose a modification of the linear model which computes instanton expansions about any limiting point in the moduli space. In the smooth case this leads to a (second) algebraic solution of the M model. We use this description to prove some conjectures about mirror symmetry, including the previously conjectured "monomial-divisor mirror map" of Aspinwall, Greene, and Morrison.

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1. Introduction

Supersymmetric nonlinear sigma models with Kähler target space form an interesting class of nontrivial field theories in two spacetime dimensions. One reason for this interest is that geometrical intuitions can be fruitfully applied in studying them. The nonrenormalization theorems [1] following from the $N = 2$ supersymmetry show that a subset of the correlation functions in these theories are not corrected from their classical values at any order in perturbation theory. The classical values can be interpreted in terms of geometrical properties of the target space and thus some of these are directly manifest in the physical theory. When the target space is a Calabi–Yau manifold the theory is in fact conformally invariant (for an appropriate choice of metric); such models comprise a large class of consistent string vacua for which the geometric interpretation yields useful insights. One striking example of this is mirror symmetry [2] which relates pairs of topologically distinct manifolds leading to isomorphic conformal field theories (but with a sign change in a certain quantum number).

The correlation functions in the distinguished set referred to above are in fact independent of the worldsheet metric. (We will follow the conventions of string theory and call the two-dimensional manifold Σ on which our field theory is defined the worldsheet.) There is a simpler version of the theory [3] which isolates these correlators, obtained by modifying the spins of the fields and performing a projection on the space of states. (This is called "twisting" the theory.) The result is a topological field theory. The correlation functions in this theory are topological invariants of the target space X corrected by instanton contributions, which themselves have a more elaborate geometrical interpretation as intersection numbers in spaces of maps $\Sigma \to X$. Thus, in principle the model is completely solved in terms of some invariants of the target space X. In practice, however, computing these rather complicated invariants of mapping spaces is a daunting task. The first few terms in the instanton expansion have been computed for a number of examples, and some progress has been made in developing more powerful tools, but in general the problem is far from solved.

When the target space is Calabi–Yau there is a choice of metric for which the sigma model is conformally invariant; in fact one then obtains invariance under the $N = 2$ superconformal algebra. Since supersymmetry transformations are generated by chiral currents, which can be either left- or right-moving on the worldsheet, there are two sets of operators whose correlators are protected by the $N = 2$ nonrenormalization theorems.

Correspondingly, there are two ways to twist the superconformal field theory to obtain a topological model. These were called the **A** and **B** model in [4]; the **A** twist, discussed in the previous paragraph, is defined for any almost complex target space. The two twists are related by mirror symmetry so that if M and W are a mirror pair of Calabi–Yau manifolds then the **A** model with target M is isomorphic to the **B** model with target W. Correlation functions in the **A** model receive instanton corrections as mentioned, while correlators in the **B** model are given exactly by their semiclassical values [5]. This has been used to replace the difficult instanton computations on M with tractable computations on W. In these applications one rewrites the sum over all instanton sectors in one model as a classical computation in the other; in this sense one has exactly summed the instanton series and produced exact correlation functions in the conformal field theory. Interpreting the coefficients in the instanton expansion in terms of the mapping spaces has thus far been the most powerful method for computing the invariants involved.

An important development in the study of these models was the discovery by Witten [6] that for a particular class of target spaces they can be obtained as the low-energy approximation to certain two dimensional $N = 2$ supersymmetric models with Abelian gauge symmetry – we will call these "gauged linear sigma models" (GLSM). This formulation led immediately to several important insights into the structure of the moduli space of these models [6]. In particular, the moduli space of the conformal field theory extends beyond the radius of convergence of the instanton expansion. Since the parameters are expressed as couplings in the GLSM Lagrangian we can directly study the model in regions beyond the radius of convergence. One finds that the low-energy dynamics in these regions does not necessarily correspond to nonlinear sigma models but can be described by other, less geometrical constructions. The different regimes in which one description or another is valid were termed "phases" in [6], and we will use this terminology. In some of these phases the low-energy theory is of a type familiar from previous studies (such as Landau–Ginzburg orbifold models) while in others we find theories which are not as well understood. An important observation is that the gauged linear model can also be twisted to obtain a topological version. (We focus on the **A** twist which is always defined; the **B** twist is only defined when certain criteria, equivalent to the requirement that the low-energy nonlinear model is conformal, are met.) Since correlation functions in this topological field theory are completely independent of the worldsheet metric they are in particular independent of scale and quantities relevant to the low-energy effective theory can be computed directly in the high-energy limit.

In this paper we will continue the program initiated in [6] of studying the topological nonlinear sigma models using the topological gauged linear model. In particular, we will study the correlation functions in the **A** twisted model. We find that as in the nonlinear case these are described in terms of a semiclassical result corrected by instanton effects. In the present case the instantons are in fact a subset of the familiar $U(1)$ instantons. A new feature of this model is that the instanton corrections can be computed exactly, and in fact the entire instanton series can be explicitly summed. The crucial difference between the GLSM computation and the nonlinear sigma model computation lies in the fact that the relevant instanton moduli spaces in the GLSM (for gauge and matter degrees of freedom) are *compact* and relatively simple spaces. As mentioned above, however, the GLSM correlation functions we compute are in fact scale-independent. Thus our computations necessarily agree with the corresponding correlators in the nonlinear sigma model which is obtained at low energies. So, in summing the series we have computed the correlation functions in the nonlinear **A** model directly. We stress that this computation does *not* involve mirror symmetry. The instanton contributions are instead computed directly, then summed. Where the mirror partner is known (or conjectured) we can compare our results to those obtained from mirror symmetry and find complete agreement.

The fields that survive the projection to the **A** twisted GLSM are twisted chiral fields. We find a rather simple description of the dynamics of these fields which in some cases allows us to obtain the exact correlation functions directly. In effect, we compute an exact expression for the effective twisted superpotential interaction in the low-energy theory. This computation is very suggestive of mirror symmetry itself, in that we obtain the exact correlators directly. In a forthcoming paper [7] we hope to make this connection explicit, obtaining a direct argument for mirror symmetry in these models. (We present some of the preliminary results of [7] at appropriate points in the present paper.) Even in cases where a direct computation of the exact result exists, we find it instructive to also derive it as an explicit sum of instanton contributions. This latter approach is at present more general, and it more directly illustrates the modification to the nonlinear sigma model instanton sum which has rendered it computable.

The topological GLSM correlation functions we compute are rational functions of the parameters. In particular, their only singularities are poles and there is no difficulty in performing analytic continuations of the series beyond their radius of convergence to the entire **A** model moduli space. Of course, this statement assumes a particular choice of the coordinates on parameter space and of a basis for the Hilbert space over every point in this

space. The choice we have made is familiar from studies of **B** models; in some sense we will have found the mirror, **A** model interpretation of this choice. We note that this is *not* the choice usually assumed in studies of the nonlinear **A** model. In particular, therefore, the coefficients in our series (which give the contributions from specific instanton sectors) are not the invariants of mapping spaces mentioned above. The two sets of computations are certainly related, as we will discuss further below, but our current methods do not suffice to extract one directly from the other.

In fact, we will find that the correlation functions can be determined from a purely algebraic computation, which is probably as near to a closed formula as one can expect to obtain. Using this formula we can check the agreement with mirror constructions. Namely, when the mirror manifold is conjecturally given by a certain construction, we can use our computation of **A** model correlation functions on one member of the pair and compare to the **B** model correlators on the other. Where we can demonstrate equality we will have gone a long way towards proving that the two are related by mirror symmetry. We will succeed in doing this for a class of models for which mirror constructions have been conjectured but not previously proved, and indicate how the extension to a much broader class of conjectured constructions could be performed.

In more detail, the structure of the paper is as follows. In section two we will quickly review the construction of the nonlinear sigma model and the twisted **A** model, and their properties. This brief review will be unnecessary for many readers; it is included to establish notation and make the work more self-contained. In section three we use the linear model to study the nonlinear topological model with target space a toric variety V. (This section begins with a brief introduction to toric geometry.) These models are interesting as solvable nonlinear sigma models. Our principal interest in this work, however, is in superconformal models related to them; in this section we will make a detailed study of the model which will lay the groundwork for subsequent developments. We find the structure of (part of) the parameter space, construct the moduli spaces of instantons, and sum the instanton series to compute the exact correlation functions. We use this formulation of the model to reproduce and clarify an algebraic construction of the solution for smooth V first proposed by Batyrev in [8]. We show how the equations obtained by Batyrev follow directly from the twisted superpotential, and also give a derivation based upon the combinatorial structure of the instanton moduli spaces (closer to the argument of [8]). In section four we modify the model, following [6], to obtain a theory whose low-energy approximation is the nonlinear sigma model with target space M a Calabi–Yau hypersurface in V. We use this

modified model to describe the structure of the moduli space. Repeating the analysis of section three we are once more able to compute the instanton corrections, sum the series, and compute the exact correlation functions. Finally, we derive an algebraic formula for the solution of the M model in terms of the V model. In section five we present a different approach to the same model, related to the nonlinear sigma model with target space V^+ the (noncompact Calabi–Yau) total space of the canonical line bundle of V. We show that in this model instanton expansions can be computed about the semiclassical limit point in any phase. Further, we rewrite the correlation functions in terms of "expectation functions" in an algebra which we construct. We then apply this formulation to proving mirror conjectures, by comparing our algebraic formula for the solution to existing algebraic formulae for **B** model correlators. For smooth V (where our algebraic solution is valid) and in the most general case for which an analogous solution to the **B** model on the conjectured mirror manifold W is known, we prove that the correlation functions in the two theories are equal. In section six we discuss some questions raised by these results and discuss open problems and directions for future work.

In most of the paper we restrict attention to a worldsheet of genus zero. The methods can be generalized to higher genus worldsheets, and we point out the generalization where appropriate. We note, however, that at higher genus the correlation functions of the twisted model will no longer coincide with those of the superconformal model.

Throughout the paper we illustrate our results by studying two particular examples, the simplest known examples of one- and two-parameter families of three-dimensional Calabi–Yau hypersurfaces in toric varieties. Each development is explicitly illustrated for these examples immediately after its introduction. We hope this format, while cumbersome, will help to clarify the discussion. We stress that the restriction to one- and two-parameter families is here one of convenience only.

This work is in a certain sense a sequel to Witten's paper [6], and in fact some of the computations for the first example discussed in this paper were performed by Witten. We thank him for sharing these with us, and encouraging us to study the generalization.

2. Précis of Nonlinear Sigma Models

Given a manifold X, equipped with a Riemannian metric g as well as a closed[1] two-form B, the nonlinear sigma model is written in terms of maps $\Phi : \Sigma \to X$ where Σ is

[1] However, see [9] for a generalization of this.

a Riemann surface. (We work with a metric of Euclidean signature on Σ throughout.) Choosing local coordinates $z, \bar{z}$ on Σ and ϕ^I on X, we write the map in terms of functions $\phi^I(z,\bar{z})$. In addition the model includes the superpartners of ϕ^I, Grassman-valued sections $\psi^I_\pm$ of $K^{\pm 1/2} \otimes \Phi^*(TX)$, where TX is the complexified tangent bundle to X and K is the canonical line bundle of Σ. In terms of these we write a component action

$$S = \int_\Sigma d^2 z \Big(\tfrac{1}{2}(g_{IJ} + iB_{IJ}) \partial_z \phi^I \partial_{\bar{z}} \phi^J + \frac{i}{2} g_{IJ} \psi^I_- D_z \psi^J_- + \frac{i}{2} g_{IJ} \psi^I_+ D_{\bar{z}} \psi^J_+ + \\ \frac{1}{4} R_{IJKL} \psi^I_+ \psi^J_+ \psi^K_- \psi^L_- \Big) \ , \tag{2.1}$$

where the covariant derivatives D are constructed by pulling back the Christoffel connection on TX. When X is complex and g is Kähler, the model has $N = 2$ supersymmetry. We assume this henceforth.

As pointed out in [3] (see [4] for the notation used here) there is a sector of this theory for which correlation functions can be computed using a greatly simplified version – the topologically twisted sigma model. This differs in the assignment of worldsheet spin to the fields. In particular, choosing local complex coordinates $\phi^i, \phi^{\bar{\imath}}$ on X we consider ψ^i_- to be a section of $\Phi^*(T^{(1,0)}X)$ (a scalar on Σ) and $\psi^{\bar{\imath}}_+$ a section of $\Phi^*(T^{(0,1)}X)$ – together they form a scalar section χ of $\Phi^*(TX)$. On the other hand, $\psi^{\bar{\imath}}_-$ becomes a $(1,0)$ form with values in $\Phi^*(T^{(0,1)}X)$ called $\psi^{\bar{\imath}}_z$ and likewise we have $\psi^i_{\bar{z}}$. In this theory one of the two supercharges under which (2.1) is invariant becomes a nilpotent global fermionic symmetry Q. It is natural to employ this as a BRST-like projection, restricting attention to Q-closed observables, i.e., those annihilated by Q. Then the nilpotency means operators of the form $\{Q, \mathcal{O}\}$ will decouple. The space of operators is thus the quotient, or the cohomology of Q. In the case at hand one finds that correlation functions of Q-closed observables will depend on the parameters g and B of the model only through the complexified Kähler class they define in $H^2(X)$. The parameter space of the model is thus naturally the space $H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})$; the quotient reflects the fact that under shifts of B by integral classes (2.1) changes by an integer so the quantum mechanical measure $e^{2\pi i S}$ is unchanged.

Explicitly, the local operators are of the form

$$\mathcal{O}_\xi = \xi_{i_1 \ldots i_s}(\Phi) \chi^{i_1} \cdots \chi^{i_s} \tag{2.2}$$

where ξ is an s-form on X. One finds

$$\{Q, \mathcal{O}_\xi\} = -\mathcal{O}_{d\xi} \tag{2.3}$$

with d the exterior derivative on X. Thus the space of operators in the topological nonlinear sigma model is the total cohomology $H^*_{\mathrm{DR}}(X)$. We can choose representatives of these classes dual to homology cycles H on X as having delta-function support on H, and denote them by $\mathcal{O}_H$.

The path integral computing a correlation function of Q-closed observables localizes on field configurations invariant under an odd symmetry [4]; in the case at hand this requires the map Φ to be holomorphic. The classical action of such a configuration is given by $S_{cl} = \int_\Sigma \Phi^*(J)$ where J is the (complexified) Kähler form on X. Further, the $N = 2$ supersymmetry ensures that the determinants of nonzero modes cancel between bosons and fermions. Thus the path integral reduces to a sum over homotopy classes of maps, weighted by $e^{2\pi i S_{cl}}$, of integrals over the finite-dimensional moduli space of holomorphic maps (instantons) in a given homotopy class. When $H_1(X, \mathbb{Z}) = 0$ we can label each instanton sector by the homology class h in $H_2(X, \mathbb{Z})$ of the image of any of the corresponding maps. The virtual dimension of the "sector h" moduli space $\mathcal{M}_h$ is $d_h = d - K \cdot h$ where K is the canonical class of X and $d = \dim_{\mathbb{C}} X$. Let us consider a given correlation function

$$\langle \mathcal{O}_{H_1}(z_1) \dots \mathcal{O}_{H_s}(z_s) \rangle \tag{2.4}$$

for $z_1 \dots z_s$ generic points on Σ and H_i homology cycles of codimension q_i. The contributions to this correlator from the various instanton sectors are referred to as *Gromov–Witten invariants*; the contribution from $\mathcal{M}_h$ vanishes unless $d_h = \sum q_i$. When nonzero the contribution is given by the integral over $\mathcal{M}_h$ of a density constructed as follows. Our choice of representatives $\mathcal{O}_H$ means the integrand has delta-function support on maps Φ such that $\Phi(z_i) \in H_i$. If $\dim_{\mathbb{C}} \mathcal{M}_h = d_h$ then maps satisfying this constraint will be isolated and the contribution is simply the number of such maps. In general, of course, $\mathcal{M}_h$ may have a dimension larger than d_h. In this case there are also zero modes of the fermions $\psi^i_{\bar{z}}$. These vary as sections of a vector bundle $\mathcal{V}$ over $\mathcal{M}_h$ of rank $\dim_{\mathbb{C}} \mathcal{M}_h - d_h$ (by the index theorem). The contribution to (2.4) is given by inserting into the integral the Euler class of $\mathcal{V}$ [10]. Eqn. (2.4) does not depend on the choice of the points z_i at which we insert the operators; we will often drop these. Summing over all possible h we obtain a (formal) series expansion for (2.4). This is expected to converge for $\mathrm{Im}(J)$ deep in the interior of the Kähler cone. Note that for such J, $\mathrm{Im}(S_{cl})$ can be made arbitrarily large for any nontrivial class h, so the series collapses in the limit onto its first term, determined by *constant* maps. The moduli space here is simply X, and the nonvanishing correlators are

the intersection numbers $\langle \mathcal{O}_{H_1}(z_1)\ldots\mathcal{O}_{H_s}(z_s)\rangle_0 = H_1\cdot H_2\cdots H_s$. The arguments of [4] ensure that for $s=2$ this is in fact the full answer; nontrivial instantons do not contribute. In particular, this guarantees that the bilinear pairing given by the two-point function is nondegenerate and a deformation invariant.

We can use the three-point functions to define a "quantum product" turning the vector space $H^*(X,\mathbb{C})$ into an associative, super-commutative algebra. We define the product $\mathcal{O}_1 * \mathcal{O}_2$ in this algebra by using the nondegenerate pairing: it is the unique element satisfying

$$\langle (\mathcal{O}_1 * \mathcal{O}_2)(z)\mathcal{O}_3(w)\rangle = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\rangle \quad \forall \mathcal{O}_3\ . \tag{2.5}$$

(Notice that the z_i in (2.5) are inserted for clarification only; nothing depends upon these; dropping them as we will often do exhibits the ring structure on H^* directly.) This algebra has some additional structure which makes it into a *Frobenius algebra*.[2] This means that the algebra – call it A – has a multiplicative identity element $\mathbb{1}$, and that there is a linear functional $\varepsilon : A \to \mathbb{C}$ such that the induced bilinear pairing $(x,y)\mapsto \varepsilon(x*y)$ is nondegenerate. There does not seem to be a standard name for such a functional; we will call it an *expectation function*. If an expectation function exists at all, then most linear functionals on A can serve as expectation functions. If A is $\mathbb{Z}$-graded, we call ε a *graded expectation function* when $\ker(\varepsilon)$ is a graded subalgebra of A (and we call A a *graded Frobenius algebra* when such a function exists). There is much less freedom to choose graded expectation functions.

The cohomology of a compact manifold X has the structure of a graded Frobenius algebra, with multiplication given by cup product, $\mathbb{1}$ given by the standard generator of $H^0(X)$, and a graded expectation function given by "evaluation on the fundamental class." The "quantum cohomology" algebra[3] of X is also a Frobenius algebra, more or less by definition: the correlator gives an expectation function, and the product (2.5) is defined with the aid of this function; the multiplicative identity is again provided by the

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[2] We follow standard mathematical usage [11,12] and do not require a Frobenius algebra to be commutative; our definition therefore differs slightly from that in [13]. However, we will primarily be interested in the even part $H^{ev}(X)$ of the cohomology of X, on which the quantum product will in fact be commutative.

[3] This is often referred to in the literature as the quantum cohomology *ring*, and indeed we will also use that terminology. However, here we use the term "algebra" in order to stress that in addition to the ring structure, this object has the structure of a complex vector space.

standard generator of $H^0(X)$. The arguments above about the limit for large J show that this quantum cohomology algebra is a deformation of the classical cohomology algebra of X, reducing to the latter for appropriate limiting values of J. This deformation however is nontrivial. In general the nilpotent, $\mathbb{Z}$-graded cohomology algebra $H^*(X)$ is deformed into an algebra which is not nilpotent and which has at most a $\mathbb{Z}_p$-grading for some integer p (when the canonical class K of X satisfies $-K = pe$ for some $e \in H^2(X, \mathbb{Z})$). In the Calabi–Yau case, however, where $K = 0$, the quantum cohomology algebra is actually $\mathbb{Z}$-graded and nilpotent.

Eqn. (2.5) shows that the correlation functions determine the ring structure. In fact they can also be used to specify a presentation of the ring in terms of generators and relations. For example, if we have a set of generators $\mathcal{O}_1, \ldots, \mathcal{O}_N$ for the even cohomology of the space X then nondegeneracy of the two-point function implies that the (even part of the) quantum cohomology ring takes the form

$$\mathcal{R} = \mathbb{C}[\mathcal{O}_1, \ldots, \mathcal{O}_N]/\mathcal{J} \tag{2.6}$$

where

$$\mathcal{J} = \{\mathcal{P} \in \mathbb{C}[\mathcal{O}_1, \ldots, \mathcal{O}_N] \mid \langle \mathcal{O}\mathcal{P} \rangle = 0 \quad \forall \mathcal{O}\} \ . \tag{2.7}$$

More generally, given any associative algebra A with multiplicative identity, and any linear functional φ on A, the kernel of the bilinear form $(x, y) \mapsto \varphi(x * y)$ is an ideal $\mathcal{J}_\varphi$, and the quotient ring $A/\mathcal{J}_\varphi$ is a Frobenius algebra with expectation function induced by φ. If A is itself a Frobenius algebra with an expectation function ε, then by a theorem of Nakayama [14] (see [12] for a modern discussion), φ takes the form $\varphi(x) = \varepsilon(\alpha * x)$ for some fixed element $\alpha \in A$, and $\mathcal{J}_\varphi$ coincides with the annihilator of α.

Although the correlation functions determine the ring structure, the opposite does not hold in general – there can be many expectation functions on a given algebra. However, if A is a graded Frobenius algebra of finite length (i.e., of finite dimension as a complex vector space) and all elements of A have non-negative degree, then the graded expectation functions on A are in one-to-one correspondence with degree 0 elements of A which are not zero-divisors. (This is because they must all be of the form $\varphi(x) = \varepsilon(\alpha * x)$ for some α which is not a zero-divisor, but every element of degree > 0 must be a zero-divisor.) In particular, in the case of the quantum cohomology algebra of a Calabi–Yau manifold X, we have a graded Frobenius algebra of finite length in which the degree 0 elements are just the one-dimensional vector space $H^0(X)$. This means that the graded expectation

function is unique up to a scalar multiple – a single normalization constant – and that the ring structure determines the correlation functions up to this overall factor. (It is not hard to see that the graded expectation function is nonzero precisely on the top degree piece $H^{2d}(X)$, where $d = \dim_{\mathbb{C}} X$, and that $H^{2d}(X)$ must also be one-dimensional.) When we have a family of rings depending on parameters, of course, the normalization can depend on these. These facts will be important in section five.

The topological field theory is obtained from the original, untwisted theory by modifying the spins of some of the fields and projecting out some of the states. As mentioned above, the fields eliminated decouple from correlators of those we keep, so their elimination has no effect on these. Furthermore, if we choose Σ so that K is trivial, the twisting itself alters nothing; this allows us to relate correlation functions computed in the topological field theory directly to correlators in the original theory, for Σ of genus zero. The mapping is described very clearly in [4].

Calabi–Yau manifolds are a special case of the above. In particular, for these we have $K = 0$ and $d_h = d$ independent of h. Thus the sum defining nonzero correlators is over all classes $h \in H_2(X, \mathbb{Z})$. This makes the model extremely hard to study, and indeed the relevant computations have thus far been essentially impossible to carry out for this class of models.[4]

The nonlinear sigma model is conformally invariant when g and B satisfy some nonlinear differential equations (which are not actually known explicitly). These can have a solution only if the target space is a Calabi–Yau manifold. When this holds, these equations should have at most one solution for each given choice of complex structure on X and a class in $H^{1,1}(X)$ to which the (complexified) Kähler class determined by g and B should belong, and there should be a solution corresponding to (g, B) whenever the Kähler class is sufficiently large. The resulting theory is then invariant under the $N = 2$ superconformal algebra. The important features of this structure, for present purposes, are the existence of chiral left- and right-moving U(1) R-symmetries, and the fact that the supercharges themselves break into left- and right-moving components. The nonrenormalization theorems protect correlators of fields annihilated by one-half of the supercharges. In the

[4] In any event, computations of even the contribution of a given instanton sector are difficult to make at any level of mathematical rigor; one example of the difficulties encountered is the fact that the instanton moduli spaces are noncompact, and the integrals – or the intersection problem they represent – must be treated with great care for boundary contributions. See [15–23] for some recent progress in this direction.

conformal case these can be chosen independently for the left- and right-moving sectors. There are thus two distinct topological sectors, and correspondingly two distinct versions of the twisted model.

The discussion up to this point has referred to the **A** model of [4]. Correlation functions in this model are independent of the complex structure; the moduli space is thus the (complexified) space of Kähler forms. The other possible twist leads to the **B** model, which is consistent only for a Calabi–Yau target. The correlation functions of the operators which survive the **B** model projection do not receive instanton corrections, and are given exactly by their semiclassical values [5]. Correlation functions in this theory are independent of the Kähler structure and the parameter space is thus the space of complex structures on X. The superconformal models exhibit mirror symmetry. This relates a Calabi–Yau manifold M to a topologically distinct "mirror manifold" W such that the nonlinear sigma models defined by the two are isomorphic. The mirror isomorphism is such that the two twists are exchanged. Thus, if M and W are mirror Calabi–Yau manifolds then the **A** model with target M is isomorphic to the **B** model with target W. The converse need not hold, i.e., the isomorphism of topological models does not guarantee an isomorphism of conformal field theories.

This property has been used to compute the quantum cohomology rings of a number of Calabi–Yau examples, as correlation functions in the **B** model are simpler objects geometrically and often computable. Translated using mirror symmetry they are interpreted as the exact sum of the instanton series for **A** model correlators. Using this interpretation, the coefficients in the expansion express geometrical properties of the moduli spaces of holomorphic maps, bypassing the technical difficulties of a direct calculation to which we alluded above. This has led to a wealth of new predictions regarding these mapping spaces, some of which have been verified. (See [15] for a review.) In principle, making these predictions requires two ingredients: a construction of the mirror manifold W given M, and an understanding of the "mirror map" between the moduli spaces of the two (isomorphic) topological field theories.

The existence of mirror manifolds, even in this weak sense of equivalence of the topological field theories, is proved only for an extremely restricted class of examples [24]. The proof uses a construction of the mirror manifold which is demonstrated to yield an isomorphic conformal field theory using the fact that at a point in the parameter space the superconformal theory is solvable. There are many conjectured generalizations of this construction [25–28] (for a review see [29]). However, no *direct* proof of mirror symmetry

has been given for any of these, and since the models are not exactly solvable, an indirect proof is difficult to construct. One of the broadest classes for which there is a conjectured mirror construction [26] is the class of all Calabi–Yau manifolds realized as hypersurfaces in toric varieties (see section three for a brief description of these objects, or [30] for an elementary introduction for physicists).

The mirror construction relates two families of conformal field theories. The members in each family are obtained by marginal deformations (of any fixed initial theory) preserving the superconformal structure. The theories are labeled by points in a "moduli space" of such deformations. Explicitly constructing the mirror isomorphism requires both a map between the two moduli spaces and a related map between the Hilbert spaces at a given point (a frame). In general, when considering families of conformal field theories there is no canonical choice of coordinates on the moduli space or of bases for the Hilbert spaces. The ambiguity is related physically to the ambiguous choice of contact terms in operator products, and geometrically to the fact that the moduli space is a nontrivial manifold and the Hilbert spaces of the theories form a nontrivial bundle over this. In twisted $N = 2$ superconformal theories, however, there is in fact a canonical choice. The reason is that the operator products of the operators which survive the projection to the topological theory are nonsingular as operators approach one another. This leads to a canonical choice of contact terms in which the insertion of multiple operators at a point is defined by point-splitting – it is the limit (as the points coincide) of insertions at distinct points. Geometrically, the existence of these coordinates means that the moduli space carries an extra structure. This structure is known as "special geometry" [31–34] when the superconformal theory has central charge 9 (corresponding to a Calabi–Yau threefold); there are natural generalizations to any dimension [35–37].

In the **A** model this canonical choice has been shown to correspond to a natural coordinate on the moduli space given by the coefficients in an expansion of the Kähler form in a fixed (topological) basis for $H^2(M)$; the preferred choice of frame is similarly given by a topological basis for $H^*(M)$. The mirror image of this in the moduli space of the **B** model on W is constructed using the periods of the holomorphic form of top degree on W using an *Ansatz* proposed in [38] and justified in [36]. When W is a hypersurface in a toric variety there is also a (geometrically) natural choice of coordinates on the moduli space of the **B** model on W, related to the coefficients of the defining polynomial. The mirror maps computed in specific examples were obtained by computing the period integrals in terms of these "algebraic" coordinates (the reason for the terminology will become clear) and using

the *Ansatz* referred to above. A conjecture for the asymptotic form of this map, termed the "monomial-divisor mirror map" was given in [39]. (Some control over asymptotics is needed because the *Ansatz* leaves a finite number of parameters undetermined.) The conjecture relates, for hypersurfaces in toric varieties, the coefficients of monomials in the defining equation directly to the expansion of the Kähler form on the mirror manifold. We will discuss this map in section five, in which we find that the GLSM naturally yields an interpretation of the algebraic coordinates in the interior of parameter space.

3. The Linear Sigma Model for V

In this section we will present a GLSM whose low-energy limit is the nonlinear sigma model with target space a toric variety V. We will then use this to completely solve the **A** model computing the exact correlation functions. For smooth V we verify an algebraic computation of these, first proposed by Batyrev [8]. This section is the longest in the paper, and by the end we will not only have solved the models in question but also developed most of the techniques we will need for the models studied in the following sections.

3.1. Toric Varieties on One Leg

A toric variety is a natural generalization of projective space. Just as $\mathbb{P}^d$ can be described in the form $\mathbb{P}^d = (\mathbb{C}^{d+1} - \{0\})/\mathbb{C}^*$, a general d-dimensional toric variety V is best thought of for our present purposes as a quotient space[5]

$$V_\Delta = (Y - F_\Delta)/T_\Delta \tag{3.1}$$

with $Y = \mathbb{C}^n$, $T_\Delta \sim \mathbb{C}^{*(n-d)}$ acting diagonally on the coordinates of Y as

$$g_a(\lambda) : x_i \to \lambda^{Q_i^a} x_i \qquad a = 1, \ldots, (n-d), \quad i = 1, \ldots, n \ , \tag{3.2}$$

and F_Δ a subset of $Y - \mathbb{C}^{*n}$ which is a union of certain intersections of coordinate hyperplanes. We often refer to (3.1) as a "holomorphic quotient" construction of V_Δ, to emphasize the holomorphic nature of the group T_Δ and its representation on Y.

---

[5] For smooth toric varieties or ones with mild singularities (so-called "simplicial" toric varieties), this is an ordinary quotient; however, in general we must take the quotient in the sense of Geometric Invariant Theory [40] (cf. also [41,42]).

The precise set of intersections of coordinate hyperplanes which constitute F_Δ, as well as the integers Q_i^a which specify the representation, are determined by the combinatorial data Δ defining V as we will describe below. The name "toric variety" refers to the algebraic torus ${\mathbb{C}^*}^d$ contained in the interior of V; the combinatorial data are in effect telling us how to (partially) compactify this torus. The fact that the torus itself is trivial means that topological (and in fact algebro-geometrical) information about V is encoded in Δ, making it easily accessible as we shall see. For details of this construction see [43,44] or [30]. In what follows we will consider the case in which V is smooth, compact, and irreducible. (Noncompact toric varieties will appear in the following sections.) We give two examples which will serve to illustrate our results in the sequel.

Example 1.

This is perhaps the simplest possible example, complex projective space $\mathbb{P}^4$. Here $Y = \mathbb{C}^5$, $F = \{0\}$, and $T = \mathbb{C}^*$ acts on the standard coordinates in Y as

$$g(\lambda) : (x_1, x_2, x_3, x_4, x_5) \mapsto (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5) \ . \tag{3.3}$$

Example 2.

The second example is chosen among other reasons because it has been studied by Candelas, de la Ossa, Font, Katz, and Morrison [45] (see also [46]). The toric variety in question is obtained by resolving the curve of $\mathbb{Z}_2$ singularities in the weighted projective space $\mathbb{P}_4^{1,1,2,2,2}$. Each point on the curve is blown up to a $\mathbb{P}^1$. We have $Y = \mathbb{C}^6$,

$$F = \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = x_5 = x_6 = 0\} \ , \tag{3.4}$$

and $T = (\mathbb{C}^*)^2$ with the action on the coordinates

$$\begin{aligned} g_1(\lambda) &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, x_2, \lambda x_3, \lambda x_4, \lambda x_5, \lambda x_6) \\ g_2(\lambda) &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (\lambda x_1, \lambda x_2, x_3, x_4, x_5, \lambda^{-2} x_6) \ . \end{aligned} \tag{3.5}$$

Note that the group element

$$g_1^2 g_2(\lambda) : (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 x_4, \lambda^2 x_5, x_6) \tag{3.6}$$

reproduces the familiar weighted projective space action on the first five variables.

We now give a somewhat more detailed version of the construction described above, explicitly describing the combinatorial structures used. This level of detail is necessary for some of the developments in section five. Let $\mathbf{N} \sim \mathbb{Z}^d$ be a lattice in $\mathbf{N}_{\mathbb{R}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{R}$. To define V we need a *fan of strongly convex rational polyhedral cones* Δ in $\mathbf{N}_{\mathbb{R}}$. This is a collection of cones with apex at the origin, each of which is spanned by a finite collection ("polyhedral") of elements in $\mathbf{N}$ ("rational") and such that the angle subtended by any two of these at the apex is less than π ("strongly convex"). To be a fan the collection must have the property that (i) any two members of the collection intersect in a common face (i.e. a cone of lower dimension bounding each) – note that this includes the origin as a possible intersection – and (ii) for each member of Δ all its faces are also in Δ. This data determines the holomorphic quotient described above as follows.

1. Let $\{v_1, \ldots, v_n\}$ be the integral generators of the one-dimensional cones in Δ, then in (3.1) we set $Y = \mathbb{C}^n$.
2. The set F_Δ is the union of intersections of coordinate hyperplanes $x_{i_1} = \cdots = x_{i_p} = 0$ for each set $1 \leq i_1 \leq \cdots i_p \leq n$ such that $v_{i_1}, \ldots v_{i_p}$ are not contained in any cone of Δ. The irreducible components of F are seen to be determined by collections as above such that any subset of $k < p$ of the vectors in the collection spans a k-dimensional cone in Δ. Following [47], these are called *primitive collections*.
3. To get T_Δ let $D \subset \mathbb{Z}^n$ be the sublattice of vectors $d = (d_1, \ldots, d_n)$ such that $\sum_i d_i v_i = 0$. Choosing a basis $\{Q^1, \ldots, Q^{n-d}\}$ for D we obtain the T action of (3.2).

Our interest in toric varieties here stems from their relation to GLSM. In mathematics, these are useful examples of nontrivial varieties whose topological and algebro-geometric properties are rather directly encoded in the combinatorics of Δ. Some examples of this will be useful in what follows. A good first example is the question whether V is compact. This will be true precisely when the fan Δ is *complete*, i.e., when the cones in Δ cover $\mathbf{N}_{\mathbb{R}}$. To see this, let $n^* \in \mathbf{N}$. To this point we can associate a $\mathbb{C}^*$ action on V as follows. Write $n = \sum_{i=1}^n k_i v_i$, where k is determined up to adding an element of the lattice D defined above. Then

$$g(\lambda) : z_i \to \lambda^{k_i} z_i \tag{3.7}$$

defines an action on V which suffers from no such ambiguity. Now consider the limit point $\lambda \to 0$. This is contained in V precisely when (i) we can choose k such that all $k_i \geq 0$; let $I \subset \{1, \ldots, n\}$ be the set for which $k_i > 0$. This means the point n^* lies in the cone spanned by the v_i, $i \in I$. The limit is then the point $z_i = 0$, $i \in I$ which is contained in

V precisely when (ii) this cone is a member of our collection Δ. In all, V will contain all such limit points precisely when any n^* is contained in some cone of Δ.

Another property encoded in Δ is the intersection theory on V (at least when V is "simplicial", which we now assume). In general, the cohomology of V is nonzero only in even dimensions; further, when V is compact $H^*(V)$ is generated by $H^2(V)$ under the intersection product. Thus the complete intersection ring is determined by relations on the elements of $H^2(V)$. These are easily read off from Δ as follows. The group $H^2(V)$ itself is generated by classes $\xi_i \quad i = 1, \ldots, n$ dual to the divisors $\{x_i = 0\}$, subject to linear relations. These essentially express the fact that T-invariant monomials in the homogeneous coordinates x_i (and their inverses) are meromorphic functions on V and hence correspond to trivial divisors. These monomials are parameterized by the lattice $\mathbf{M} \sim \mathbb{Z}^d$ dual to $\mathbf{N}$, since 3. above guarantees that for $m \in \mathbf{M}$ the monomial $\chi_m = \prod_{i=1}^n x_i^{\langle m, v_i \rangle}$ is T-invariant. Thus we have

$$\sum_{i=1}^{n} \langle m, v_i \rangle \xi_i = 0 \tag{3.8}$$

for every $m \in \mathbf{M}$. There are d independent relations of this type, which reduce the dimension of $H^2(V)$ to $n-d$. This determines a basis η_a of $H^2(V)$ in which we can write

$$\xi_i = \sum_{a=1}^{n-d} Q_i^a \eta_a \ . \tag{3.9}$$

There are also nonlinear relations in the ring $H^*(V)$. These are read off most easily in the dual picture as excluded intersections of the coordinate hyperplanes. That is, for each irreducible component of F, described as $\{x_a = 0 \mid a \in A\}$ for some set $A \subset \{1, \ldots, n\}$, we get a relation

$$\prod_{a \in A} \xi_a = 0 \ . \tag{3.10}$$

(These relations comprise what is known as the *Stanley–Reisner ideal.*)

The relations (3.9) and (3.10) determine the ring structure of $H^*(V)$ completely; the one thing left undetermined is the normalization of the expectation function $\langle \ \rangle_V$ given by evaluation on the fundamental class of V. This can also be determined by the toric data, as follows. Given a collection of d distinct coordinate hyperplanes $\{x_{i_1} = 0\}$, $\ldots$, $\{x_{i_d} = 0\}$ which *do* intersect on V, we have

$$\langle \xi_{i_1} \ldots \xi_{i_d} \rangle_V = \frac{1}{\mathrm{mult}(v_{i_1}, \ldots, v_{i_d})} \ , \tag{3.11}$$

where $\mathrm{mult}(v_{i_1}, \ldots, v_{i_d})$ denotes the index in $\mathbf{N}$ of the lattice spanned by these vectors. (This index is always 1 if V is smooth.)

Example 1.

Here there is precisely one η, and (3.9) becomes $\xi_i = \eta \quad \forall i$. From the form of F we have $\eta^5 = 0$. The one-dimensional cones in Δ are generated by

$$\begin{aligned} v_1 &= (-1,-1,-1,-1) \\ v_2 &= (1,0,0,0) \\ v_3 &= (0,1,0,0) \\ v_4 &= (0,0,1,0) \\ v_5 &= (0,0,0,1)\ , \end{aligned} \tag{3.12}$$

and (3.11) yields $\langle \eta^4 \rangle_V = 1$.

Example 2.

Here the linear relations read

$$\begin{aligned} \xi_1 &= \xi_2 = \eta_2 \\ \xi_3 &= \xi_4 = \xi_5 = \eta_1 \\ \xi_6 &= \eta_1 - 2\eta_2 \end{aligned} \tag{3.13}$$

and the nonlinear relations are

$$\begin{aligned} \xi_1 \xi_2 &= \eta_2^2 = 0 \\ \xi_3 \xi_4 \xi_5 \xi_6 &= \eta_1^3 (\eta_1 - 2\eta_2) = 0 \end{aligned} \tag{3.14}$$

(compare (3.4)). The one-dimensional cones in Δ are spanned by

$$\begin{aligned} v_1 &= (-1,-2,-2,-2) \\ v_2 &= (1,0,0,0) \\ v_3 &= (0,1,0,0) \\ v_4 &= (0,0,1,0) \\ v_5 &= (0,0,0,1) \\ v_6 &= (0,-1,-1,-1)\ , \end{aligned} \tag{3.15}$$

and (3.11) yields $\langle \eta_1^4 \rangle_V = 2 \langle \eta_1^3 \eta_2 \rangle_V = 2$.

There is an equivalent construction of V, in which we consider Y as a symplectic manifold (with the standard symplectic form $\omega = i\sum_{i=1}^{n} dz^i \wedge d\bar{z}^{\bar{\imath}}$), with a symplectic action by the maximal compact subgroup $G \subset T$. In our case $G = U(1)^{(n-d)}$ acts as in (3.2) with $|\lambda| = 1$. The *symplectic reduction* of Y by G depends on a choice of "moment map" $\mu : Y \to (\mathrm{Lie}(G))^{\vee}$. In coordinates, the components of $\mu : Y \to \mathbb{R}^{(n-d)}$ are simply generators (by Poisson brackets) of the G action, and can be described by

$$\mu_a = \sum_{i=1}^{n} Q_i^a |x_i|^2 - r_a \ , \tag{3.16}$$

where r_a are undetermined additive constants. The symplectic reduction is then defined as

$$V(r) \equiv \mu^{-1}(0)/G \ . \tag{3.17}$$

The structure of $V(r)$ depends on r, of course. Its topology and even its dimension will change as r varies. From the form of (3.16) it is clear that every T-orbit in Y will contribute at most one point to $V(r)$ (or one G-orbit to $\mu^{-1}(0)$). Which orbits will contribute is determined by the value of r – these are the T-orbits which intersect $\mu^{-1}(0)$. There is a cone in r-space for which the set of T-orbits which do not contribute is precisely F_Δ. For these values of r we obtain a manifold topologically identical to V as defined by (3.1). The quotient space $V(r)$ inherits a symplectic form ω_r by reducing ω. (That is, the restriction of ω to $\mu^{-1}(0)$ is G-invariant, and so induces a symplectic form on V.) The cohomology class of ω_r in $H^2(V,\mathbb{R})$ depends linearly on r (for r in the cone corresponding to V).[6] The coefficients of r_a are essentially the η_a of (3.9). The symplectic reduction carries a natural complex structure, in which the reduced symplectic form becomes a Kähler form. The range of allowed r is in fact precisely the Kähler cone of V [50]; for r in this cone the two constructions are identical. The symplectic construction makes sense for *any* value of r. However, it does not necessarily produce a smooth space or one of dimension d; there will in general be values of r for which the space $V(r)$ is altogether empty. We denote $\mathcal{K}_c$ the $(n-d)$-dimensional cone of values of r for which $V(r)$ is nonempty. This is the cone spanned by the vectors Q_i, and in general is larger than the Kähler cone $\mathcal{K}_V$. However, for a compact V, the cone $\mathcal{K}_c$ is always convex, since otherwise we would have a combination $\sum_i a_i Q_i^a = 0$ for some non-negative integers a_i not all zero; then $\prod_i x_i^{a_i}$ would be a nonconstant holomorphic function on V.

[6] This was shown in [48] to follow from [49].

3.2. The Lagrangian

We are interested in studying the nonlinear sigma model with target space V. As is the case for the well-known example of $\mathbb{P}^n$, this is related to an $N = 2$ supersymmetric gauged linear sigma model with target space Y and gauge group $G = U(1)^{(n-d)}$ such that $G_{\mathbb{C}} = T$.

This model contains vector superfields V_a with component expansions[7]

$$V = -\sqrt{2}(\theta^-\bar{\theta}^- v_{\bar{z}} + \theta^+\bar{\theta}^+ v_z - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma}) + i(\theta^2\bar{\theta}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}} - \bar{\theta}^2\theta^\alpha\lambda_\alpha) + \tfrac{1}{2}\theta^2\bar{\theta}^2 D \ , \tag{3.18}$$

and chiral superfields Φ_i with component expansions

$$\Phi = \phi + \sqrt{2}(\theta^+\psi_+ + \theta^-\psi_-) + \theta^2 F + \cdots \ , \tag{3.19}$$

where $\cdots$ are total derivative terms, as well as their complex conjugates $\overline{\Phi}$. We couple n chiral matter multiplets Φ_i with charges Q_i^a under G to the $n-d$ abelian gauge superfields V_a, and introduce Fayet-Illiopoulos terms for the abelian gauge symmetry. In the limit of infinite coupling the gauge fields act as constraints, and the action may be written in superspace as

$$S = \int_\Sigma d^2z\, d^4\theta \left[\sum_{i=1}^{n} \overline{\Phi}_i \exp\left(2\sum_{a=1}^{n-d} Q_i^a V_a \right) \Phi_i - \sum_{a=1}^{n-d} r_a V_a \right] \ . \tag{3.20}$$

The V_a can be eliminated by their equations of motion leading to a nontrivial kinetic term for the matter fields corresponding to the analog of the Fubini–Study metric on V. (Expanding in components we reproduce (2.1).)

For the present application it is perhaps better to start off with nonzero kinetic terms for the gauge fields (these will be generated dynamically in any case, so this is more a shift of perspective than a change in the model). We thus add to (3.20) a kinetic term for the gauge fields, and explicitly allow for the inclusion of θ angles, so the total action is

$$\begin{aligned} S = \int_\Sigma d^2z\, d^4\theta \left[\sum_{i=1}^{n} \overline{\Phi}_i \exp\left(2\sum_{a=1}^{n-d} Q_i^a V_a \right) \Phi_i - \sum_{a=1}^{n-d} \frac{1}{4e_a^2} \overline{\Sigma}_a \Sigma_a - \sum_{a=1}^{n-d} r_a V_a \right] + \\ \int_\Sigma d^2z \sum_{a=1}^{n-d} \frac{\theta_a}{2\pi i} f_a \ , \end{aligned} \tag{3.21}$$

[7] The conventions are those of [6]; we work in WZ gauge.

where f_a is the curvature of the gauge connection, and $\Sigma_a = \frac{1}{\sqrt{2}}\overline{D}_+ D_- V_a$ is the (twisted chiral) gauge-invariant field strength associated to the gauge field V_a with component expansion

$$\Sigma = \sigma - i\sqrt{2}(\theta^+ \bar{\lambda}_+ + \bar{\theta}^- \lambda_-) + \sqrt{2}\theta^+ \bar{\theta}^- (D - f) + \cdots \ . \tag{3.22}$$

The last two terms in (3.21) can be rewritten as

$$S_{D,\theta} = \sum_{a=1}^{n-d} \int d^2 z(-r_a D_a + \frac{\theta_a}{2\pi i} f_a) = \int d^2 z d\theta^+ d\bar{\theta}^- \widetilde{W}(\Sigma)|_{\theta^- = \bar{\theta}^+ = 0} + \text{c.c.} \ , \tag{3.23}$$

where

$$\widetilde{W}(s) = \frac{i}{2\sqrt{2}} \sum_{a=1}^{n-d} \tau_a s_a \tag{3.24}$$

with

$$\tau_a = i r_a + \frac{\theta_a}{2\pi} \ . \tag{3.25}$$

This interaction is a twisted superpotential for the twisted chiral fields Σ_a. As we shall see, there are regions in which these are the low-energy degrees of freedom. Then integrating out the massive chiral fields we will obtain an effective action for Σ_a. As is familiar with chiral superpotentials, this effective (twisted) superpotential is constrained by the requirements of holomorphy in the fields Σ and in the couplings τ.[8] In some cases this property, together with computable limiting properties, will suffice to yield an exact expression.

The action (3.21) is invariant under a large group of global symmetries, in addition to the local G symmetry. Of special importance is the existence of left- and right-moving $U(1)$ R-symmetries. We can choose these so that under the left-moving R-symmetry $U(1)_L$ the charged fields are $(\psi^i_-, F^i, \sigma_a, \lambda^-_a)$ with charges $(-1,-1,-1,1)$ (and of course their complex conjugates with opposite charges), while under the right-moving R-symmetry $U(1)_R$ the charged fields are $(\psi^i_+, F^i, \sigma_a, \lambda^+_a)$ with charge $(-1,-1,1,1)$. These symmetries suffer from gauge anomalies, given by

$$\Delta(Q_R) = -\Delta(Q_L) = -\frac{1}{2\pi} \sum_{i=1}^{n} \left(\sum Q_i^a \int_\Sigma d^2 z \, f_a \right) \ . \tag{3.26}$$

The nonchiral combination $Q_L + Q_R$ is conserved. In addition there is a group $H = U(1)^{n-d}$ of nonanomalous chiral global symmetries, acting by phases on Φ_i. This is the ungauged subgroup of the full $U(1)^n$ phase symmetry.[9]

[8] A holomorphic dependence on the couplings in a two-dimensional theory is familiar. For a recent, more general application in four dimensions see [51].

[9] This is the full symmetry group in generic cases. When the charges Q_i^a are degenerate or satisfy appropriate divisibility conditions the group is actually larger. In our two examples the groups are $U(5)$ and $U(2) \times U(3) \times U(1)$ respectively.

3.3. The Low-Energy Limit

As mentioned above, this model is expected to reduce to the nonlinear sigma model with target space V in the infrared (at strong coupling). We can see this explicitly by solving for the auxiliary fields D_a (setting all the gauge couplings equal)

$$D_a = -e^2 (\sum_{i=1}^{n} Q_i^a |\phi_i|^2 - r_a) \ . \tag{3.27}$$

Comparing to eqn. (3.16) we see that $D_a = -e^2 \mu_a$ are just the components of the moment map to within an irrelevant constant.

We are interested in finding the space of classical ground states of the theory. To this end we consider the potential for the bosonic fields

$$U = \sum_{a=1}^{n-d} \frac{(D_a)^2}{2e^2} + 2 \sum_{a,b=1}^{n-d} \overline{\sigma}_a \sigma_b \sum_{i=1}^{n} Q_i^a Q_i^b |\phi_i|^2 \ . \tag{3.28}$$

Setting $U = 0$ we obtain the restriction to $D^{-1}(0)$. For values of r in the appropriate range (see below) solving $D = 0$ will lead to expectation values for the ϕ_i which break G by the Higgs mechanism[10] to a discrete subgroup, and lead to a nondegenerate mass matrix for the complex scalars σ whose expectation values thus vanish. When this obtains we see that the manifold of gauge-inequivalent vacua is indeed

$$D^{-1}(0)/G = V(r) \ . \tag{3.29}$$

The massless modes (classically) will be oscillations of ϕ, ψ tangent to V. Thus, when r lies in the Kähler cone of V we obtain the nonlinear sigma model with target space V as the low-energy limit. The discussion in the previous subsection then identifies the τ coordinates with the canonical coordinates on the moduli space of Kähler structures on V, since the class of the Kähler form ω_r on the symplectic reduction $V(r)$ will depend linearly on r.

What happens when r lies outside this cone? For r outside the cone $\mathcal{K}_c$ defined above, the equation $D = 0$ has no solutions and (3.28) suggests that supersymmetry is broken.

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[10] The attentive reader will wonder at the appearance of a Higgs phase, since the model is usually thought to exhibit confinement. In fact the question is moot. Gauge-invariant correlation functions cannot settle the confinement issue. Certainly the Higgs description is valid at weak coupling. We thank E. Rabinovici and N. Seiberg for discussions of this point.

We will find later that this is not the case (in general the theory has a nonzero Witten index so this is not too surprising). For now, we restrict attention to $r \in \mathcal{K}_c$. This is still larger than the Kähler cone, so there are values of r for which we cannot expect the model to reduce to the nonlinear sigma model with target space V at low energies. In general, the classical analysis predicts the existence of a set of codimension-one cones on which the model is singular. The singularities occur whenever there are solutions to $D = 0$ which leave a continuous subgroup of G unbroken. With ϕ set to one of these solutions, the σ field associated to this subgroup has a flat potential and the space of classical ground states is noncompact. The form of (3.27) shows that the subgroup generated by g_a can be unbroken when $D = 0$ is consistent with $\phi_i = 0$ for all i such that $Q_i^a \neq 0$. This in turn constrains r to lie in a codimension-one cone.

The singular loci will divide $\mathcal{K}_c$ into components, one of which will be the Kähler cone $\mathcal{K}_V$. The low-energy theory in other components will have a different interpretation. An equivalent characterization of the singularities is as those values of r at which some component of F disappears. On the two sides of the singular locus the sets F differ, leading to topologically distinct quotients $V(r)$. Several possible types of models can be encountered here. The first possibility is that the quotient space is a smooth manifold. In this case the considerations of the previous paragraph hold and the manifold in question is a toric variety. There can be any number of such topologically distinct quotients in the parameter space of a model; they are related by birational transformations. Another possibility is that singular quotient spaces arise. In this case the holomorphic interpretation is not as clear. In particular, the interpretation of r as coefficients in an expansion of the Kähler class requires some generalization here (see [39]). In summary, the classical analysis predicts the existence of several "phases" [6] of the model, separated in r-space by singularities. We will refine this picture in what follows.

Example 1.

For $\mathbb{P}^4$, we have $D = -e^2(\sum_{i=1}^{5} |\phi^i|^2 - r)$. Thus we see that $\mathcal{K}_c = \{r > 0\} = \mathcal{K}_V$; in this region the quotient space is just V.

Example 2.

In the second example we have

$$\begin{aligned} D_1 &= -e^2(|\phi_3|^2 + |\phi_4|^2 + |\phi_5|^2 + |\phi_6|^2 - r_1) \\ D_2 &= -e^2(|\phi_1|^2 + |\phi_2|^2 - 2|\phi_6|^2 - r_2) \ . \end{aligned} \tag{3.30}$$

We see that $\mathcal{K}_c$ is given by the inequalities

$$\begin{aligned} r_1 &> 0 \\ 2r_1 + r_2 &> 0 \ . \end{aligned} \tag{3.31}$$

In this cone there is a ray (codimension-one cone) $r_2 = 0,\ r_1 > 0$ on which we can have $\phi_1 = \phi_2 = \phi_6 = 0$ solving $D = 0$. When this obtains we see from (3.5) that g_2 is unbroken; hence σ_2 is free. The existence of this noncompact component of field space means that the theory is singular. This singularity separates the first quadrant from the cone $r_2 < 0,\ 2r_1 + r_2 > 0$. In the first quadrant (3.30) shows that the excluded set is precisely (3.4). This region is thus the Kähler cone $\mathcal{K}_V$. For r in the other cone in $\mathcal{K}_c$, we see from (3.30) that the excluded set is $F = \{\phi_6 = 0\} \cup \{\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0\}$. Thus the quotient space is topologically distinct from V. In some sense this phase corresponds to the original (unresolved) weighted projective space. (Following [30], we refer to this as the "orbifold" phase since the space of classical vacua has orbifold singularities.) In the holomorphic version this means that since ϕ_6 is nonzero we can use g_2 to fix it (say at $\phi_6 = 1$); the remaining symmetry $(g_1^2 g_2)$ is the expected $\mathbb{C}^*$ action on the homogeneous coordinates. Figure 1 shows this structure in r space.

3.4. Singularities and Quantum Corrections

The classical analysis of the preceding subsections led us to the conclusion that for $r \in \mathcal{K}_V$ the GLSM is equivalent in the low-energy limit to a nonlinear sigma model with target space V, determined by the Kähler form obtained via symplectic reduction. In terms of a fixed topological basis for $H^2(V)$ this is proportional to $\omega = \sum_{a=1}^{n-d} r_a \eta_a$, and the cone $\mathcal{K}_V$ is the Kähler cone. The B-field on V is likewise linearly related to the θ angles. In terms of the canonical coordinates introduced in section two this classical result is

$$t_a = \tau_a \ . \tag{3.32}$$

This is expected to approximate the exact answer for values of r deep in the interior of the cone $\mathcal{K}_V$, because there the model is weakly coupled (from our normalization of the gauge multiplet it follows that $1/r_a$ is the coupling constant for the ath factor of G) and the classical analysis should be approximately valid.

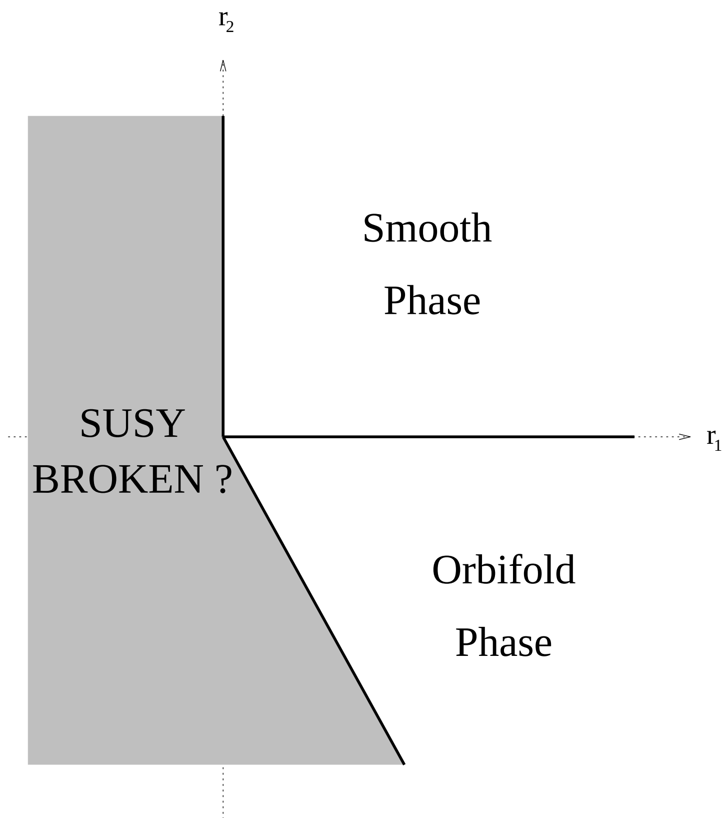

Figure 1. Classical phase diagram.

As we decrease the value of r we will find quantum corrections to these results. There is a natural deformation of the low-energy model (given by deforming the Kähler class) which one expects to correspond to the deformation of the ultraviolet theory by changing r. Thus we can expect that at least for sufficiently large r (we use this language somewhat loosely; what we mean is r deep in $\mathcal{K}_V$, so that all of the gauge couplings are small) the low-energy theory is given by the nonlinear sigma model with target space V and some Kähler form. The quantum corrections then take the form of corrections to the simple relation between the parameters of the low-energy theory and those of the original model. The corrections are constrained by $N = 2$ nonrenormalization theorems as follows. Correlation functions which are holomorphic in τ in the microscopic theory (correlators of twisted chiral fields – see the next subsection) will map to correlation functions holomorphic in t. This implies that $t(\tau)$ is holomorphic. Thus the perturbation series for $t = \tau + a_1 + a_2 r^{-1} + \cdots$ must terminate at a_1, since higher-order terms are not analytic in τ. Thus we expect that to all orders in $1/r$, (3.32) is modified at most by the addition of a constant term. In addition there can be nonperturbative corrections which drop off exponentially at large r. These corrections are of course important in our application of the GLSM to a study of the

low-energy nonlinear model. We will find it convenient to compute them rather indirectly, using the properties of instanton corrections to the correlation functions. In the models of this section, the corrections will be found to vanish and in fact (3.32) holds *exactly*.

As r approaches the boundaries of $\mathcal{K}_V$ the classical theory is singular signaling a possible breakdown of the approximation. The singularity arises from the region in field space in which the scalars σ are large. We now turn to a more careful study of this region. We will find that some of the singularities predicted by the classical analysis are true singularities while others are removed by quantum corrections. Further, we will find as expected that supersymmetry breaking does not occur and explicitly find the vacuum structure outside the cone $\mathcal{K}_c$.

Let us then consider the large-σ region in field space. In this region the scalars φ_i are massive and their expectation values vanish because of the second term in (3.28). Then G is unbroken and the theory is approximated by free $N = 2$ gauge theory, weakly coupled to heavy matter fields. In this theory, as is well-known, the Fayet-Illiopoulos term breaks supersymmetry at the classical level (eqn. (3.28) gives $U \geq \sum_a r_a^2/2e^2$). In fact, as one would expect by holomorphy, the θ angles, which we have ignored thus far, also break supersymmetry. These are equivalent to a constant background electric field [52] and the ground-state energy is

$$U(r,\theta) = \frac{e^2}{2} \sum_{a=1}^{n-d} \left(r_a^2 + \left(\frac{\hat{\theta}_a}{2\pi} \right)^2 \right) , \tag{3.33}$$

where $\hat{\theta}_a = \theta_a + 2\pi m_a$ for an integer m_a such that (3.33) is minimized. Physically, this shift reflects the fact that a field corresponding to $|\theta| > \pi$ will be screened by pair creation.

We have neglected the chiral fields. Integrating them out will lead to small corrections suppressed by their large mass (of order $|\sigma|$), unless they appear in the loops of divergent graphs. In fact, there is precisely one relevant divergence [6] – the one-loop correction to the expectation value of the auxiliary field D. This arises through boson loops; the σ-dependence enters through the effective mass of these bosons. Thus[11]

$$\frac{1}{e^2} \langle D_a \rangle_{1-\mathrm{loop}} = \sum_{i=1}^{n} Q_i^a \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + 2Q_i^a Q_i^b \bar{\sigma}_a \sigma_b} . \tag{3.34}$$

[11] We thank E. Witten for pointing out to us that the corresponding formula in [6] needed a correction, as given here.

Cutting off the divergent integral we have

$$\frac{1}{e^2}\langle D_a\rangle_{1-\mathrm{loop}} = \frac{1}{4\pi}\sum_{i=1}^{n} Q_i^a \log\left(\frac{\Lambda^2}{2Q_i^aQ_i^b\bar{\sigma}_a\sigma_b}\right) \ . \tag{3.35}$$

The correction can be interpreted as a σ-dependent shift in r. There is also a corresponding shift in θ, verified by computing the one-loop correction to $\langle\lambda_+\bar{\lambda}_-\rangle$. We incorporate both of these in a perturbative correction to the twisted chiral superpotential for Σ (3.23), valid for large σ

$$\widetilde{W}(\Sigma) = \frac{1}{2\sqrt{2}}\sum_{a=1}^{n-d}\Sigma_a\left(i\hat{\tau}_a - \frac{1}{2\pi}\sum_{i=1}^{n} Q_i^a \log(\sqrt{2}\sum_{b=1}^{n-d} Q_i^b\Sigma_b/\Lambda)\right) \ . \tag{3.36}$$

This corrected twisted superpotential leads to a modification in the potential energy

$$U(\sigma) = \frac{e^2}{2}\sum_{a=1}^{n-d}\left|i\hat{\tau}_a - \frac{\sum_{i=1}^{n} Q_i^a}{2\pi}(\log(\sqrt{2}\sum_{b=1}^{n-d} Q_i^b\sigma_b/\Lambda)+1)\right|^2 \ . \tag{3.37}$$

The computation we have performed is valid at large values of σ. One observes that deep in the interior of $\mathcal{K}_V$ (3.37) will vanish (indicating new vacua missed by the classical analysis) for small σ. Here the approximation fails; more precisely, these classical vacua are unstable. The true vacuum states are found at large ϕ and $\sigma = 0$ as described in the previous subsection. This can change at the boundaries of this cone, where singularities were predicted classically. These singularities arise precisely from the existence of low-energy states at large σ. We first consider values of r in the vicinity of the classical singularity, and very far from the origin in r-space. In this region the classical description is a good approximation for most of the theory, the exception being the strongly-coupled dynamics of the unbroken gauge symmetry, which we relabel V_1. The fields σ_a for $a \neq 1$ are massive; the low-energy degrees of freedom are the chiral fields neutral under g_1 and the gauge multiplet V_1. The two sectors are weakly coupled by the massive chiral fields charged under g_1. The effective twisted superpotential for Σ_1 after integrating out these massive chiral fields can be computed by setting $\sigma_a = 0$ for $a > 1$ in (3.36), in which case the potential (3.37) reduces to

$$U(\sigma_1) = \frac{e^2}{2}\left|i\hat{\tau}_1 - \frac{\sum_{i=1}^{n} Q_i}{2\pi}(\log(\sqrt{2}Q_i\sigma_1/\Lambda)+1)\right|^2 \ , \tag{3.38}$$

where $Q_i \equiv Q_i^1$. The effect of this correction on the physics differs dramatically, depending on whether or not the equality

$$\sum_{i=1}^{n} Q_i = 0 \tag{3.39}$$

holds.

Let us first address the situation in which (3.39) is satisfied. Then (3.38) reduces to

$$U(\sigma_1) = \frac{e^2}{2} \left| i\hat{\tau}_1 - \frac{1}{2\pi} \sum_{i=1}^{n} Q_i \log(Q_i) \right|^2 \ . \tag{3.40}$$

Indeed in this case the integral (3.34) is convergent, and the correction to τ is in fact finite and σ-independent

$$\tau_{\rm eff} = \tau + \frac{i}{2\pi} \sum_{i=1}^{n} Q_i \log(Q_i) \ . \tag{3.41}$$

This is in accord with the fact that instantons in this factor of G do not contribute to the anomaly. The linear twisted superpotential is the unique form that does not break the symmetry explicitly. Since the correction in σ-independent we will assume that it holds away from the region (large-σ_1) in which the computation is valid. For a particular value of θ_1 (0 or π) (3.40) approaches zero as $r_1 \to 0$, leading to the singularity in the low-energy theory discussed earlier. For other θ_1, however, there is no singularity; the minimum energy at which the large-σ_1 region of field space becomes accessible is nonzero, the space of supersymmetric ground states is compact and the theory nonsingular. Thus the singularities occur in codimension-two subspaces of parameter space. This will be very important in what follows. In particular, it means that we can continuously deform the theory "around" the singular loci to connect models in different "phases" [6,30]. We stress that (3.41) is valid *only* in the vicinity of one of the asymptotic components of the singular locus (i.e. r_1 small and all other r_a large). It is useful for predicting the location of the singularities far from the origin in r-space. It should not, however, be confused with perturbative corrections to (3.32). We will compute these in the sequel.

Things are very different if (3.39) is not satisfied. In this case the integral (3.34) is divergent, quantum corrections are large, and we can expect qualitative modifications to our classical conclusions. Indeed, in this case (3.38) grows as $|\log(\sigma_1)|^2$ at large σ_1, so the field space accessible to very low-lying states is effectively compact and there is

no singularity for any value of τ_1.[12] On the other hand, continuing past the nonexistent "singularity" to a region in which r_1 differs in sign from $\sum_i Q_i$, we now find at large $|r_1|$ new vacuum states. In these states the expectation values of ϕ_i are restricted to those values invariant under g_1 (hence leaving σ_1 massless), and the expectation value of σ_1 is determined by the requirement that (3.38) vanish. These values are given by

$$\sigma_1 = \frac{\Lambda}{e} \exp\left[\frac{2\pi i\hat{\tau}_1}{\sum_i Q_i}\right] . \tag{3.42}$$

Note that for large $|r_1|$ these values lie at large $|\sigma_1|$, justifying the use of (3.38). There will be $\sum_i Q_i$ of these because of the freedom to shift $\hat{\tau}_1$ by an integer. The boundaries of $\mathcal{K}_c$ are necessarily of this type, and the analysis above shows that supersymmetry is not broken for values of r outside this cone, correcting the classical prediction.

In all, we have been led to modify our classical analysis as follows. First, the regions outside the cone $\mathcal{K}_c$ are in fact continuously connected to the interior; there is no supersymmetry breaking. However, in these regions the physical theory has additional vacuum states involving the gauge multiplet in a nontrivial way and cannot be expected to reduce to a nonlinear sigma model at low energies. Furthermore, some of the regions inside $\mathcal{K}_c$ may be of this type. This will be the case if in reaching them from the Kähler cone we have crossed a boundary for which (3.39) does not hold. The models for which a (standard) geometrical interpretation is expected will occupy a smaller cone $\mathcal{K}_q$; this is the cone of r's for which the corresponding symplectic reduction $V(r)$ is a space of complex dimension d. Finally, our naïve predictions for the singularities of the model are corrected. The true singular locus is, as we shall see, quite complicated in general. What we have found is that far from the origin it asymptotically approaches the cones in r predicted classically, shifted as in (3.41). At each of these the singularity is restricted to a particular value of θ. The fact that the singularity occurs in complex codimension one (consistent with the holomorphic properties of these models) is important; it means one can connect two points lying in different regions in r-space without encountering a singularity. Despite this fact, we will follow [6] and refer to the theories in regions of r-space separated by singularities as different phases of the model.

[12] There will always be one such symmetry in G if V is compact. This corresponds to the fact that the semipositive anticanonical class is nonzero for a compact toric variety V. It is also in accord with the existence of an anomalous R-symmetry, which means that one θ angle could be eliminated by field redefinitions.

Example 1.

The singularity at $r = 0$ does not satisfy (3.39). We conclude that there is in fact no singularity, and that the $r < 0$ theory exists but is not described by a nonlinear sigma model. A different interpretation of the analytic continuation of this model was proposed in [53].

Example 2.

The singularity at $r_2 = 0$ does satisfy (3.39). There is thus a true singularity (at $\theta = 0$; we can of course choose a path in parameter space connecting the two "phases" that does not meet this singularity) and it occurs at $\tau_2 = i\frac{\log 2}{\pi}$. In this example $\mathcal{K}_q = \mathcal{K}_c$. The singularities at the boundary of $\mathcal{K}_q$ do not satisfy (3.39) and are thus not true singularities.

3.5. Determining the Singular Locus

The analysis in the previous subsection has corrected the classical predictions for the singularities of the model far from the origin. The exact singular locus of the model diverges from the semiclassical predictions due to the effects of instantons in the Higgs sector of the theory, and interpolates between the asymptotic pieces computed above. To find this locus would apparently require computations in the strongly-coupled interior of r-space near the origin. This appearance is misleading, however, because in fact the twisted superpotential (3.36) is an exact expression. Perturbative corrections at higher order are of course precluded by holomorphy in τ. We claim, however, that nonperturbative corrections are absent as well. This will be borne out in the following subsections by explicit computations of the exact correlation functions and a study of their singularities.[13] We defer a complete argument for the vanishing of these corrections to [7].

The equations of motion for σ which follow from the twisted superpotential interaction

$$\frac{\partial \widetilde{W}}{\partial \sigma_a} = 0 \ , \tag{3.43}$$

[13] Similar methods were used in [54] to compute exact superpotentials in four dimensions. N. Seiberg has recently presented an argument for the assumption made in this work (and justified by its results as we do here) that the exact effective superpotential is linear in the tree-level coupling.

can be rewritten using (3.36) as

$$\prod_{i=1}^{n}\left(\frac{\sqrt{2}e}{\Lambda}\sum_{b=1}^{n-d}Q_i^b\sigma_b\right)^{Q_i^a}=e^{2\pi i\tau_a}\ ,a=1,\ldots,n{-}d\ . \tag{3.44}$$

These equations will lead to some exact properties of the correlation functions of the σ_a in what follows, but we first show how they lead directly to the exact singular locus. This happens because the singularity, as observed above, arises from the existence of solutions to (3.44) (nontrivial vacuum states) at large σ. Let us choose a basis for G such that $\sum_{i=1}^{n}Q_i^a=Q\delta^{a1}$ for some constant Q, so that (3.39) is satisfied for $a>1$. Then we have seen that the large-σ_1 region of field space does not lead to a singularity; the "geometric" phases of the theory are then in the region (say) $r_1>0$, and for the moment we concentrate on this region. Then the field σ_1 is massive and we integrate it out to obtain the low-energy effective action. We will in fact simply set $\sigma_1=0$ and drop it, an approximation which should be valid for sufficiently large r_1. At generic large values of the other σ_a the chiral fields which remain massless are the set $\{\phi_i\}_{i\in I^c}$ of fields satisfying $Q_i^a=0$ for $a>1$. Expectation values for these do not break the subgroup $H\subset G$ generated by g_a for $a>1$. The remaining chiral fields are massive and will be integrated out as well. This will lead to an effective twisted superpotential for the massless σ_a, yielding as equations of motion

$$\prod_{i\in I}\left(\sum_{b=2}^{n-d}Q_i^b\sigma_b\right)^{Q_i^a}=e^{2\pi i\tau_a}\ ,a=2,\ldots,n{-}d\ , \tag{3.45}$$

where we have used (3.39) to eliminate the Λ-dependent constant.

We now note that the equations (3.45) are all homogeneous (of degree zero) in σ because of (3.39). This homogeneity means two things. First, if the equations have a solution at all there are solutions at arbitrarily large σ and the model is indeed singular. Further, because (3.45) can be interpreted as $n{-}d{-}1$ equations for the $n{-}d{-}2$ *ratios* of the nonzero σ_a's, the equations are overdetermined. They can be considered the parametric equations of some subvariety of τ-space of dimension $n{-}d{-}1$ (recall τ_1 is unconstrained), along which the models are singular.

This subvariety (a part of the "singular locus" in the parameter space) will have several components at large-r, and by inspection these can be seen to coincide with some of the singularities in this region predicted semiclassically above. In these limits the solutions indeed tend to the limiting direction in σ-space which was predicted from the classical

action. In the interior of r-space, the true singular locus interpolates smoothly between these various limits. At a generic point on it the σ vacua are in a generic direction in σ-space (generic, that is, among σ's constrained by $\sigma_1 = 0$), leading to nonzero masses for all of the chiral fields $\phi_i,\ i \in I$. We have made a computation at large r_1 to describe this part of the singular locus, but note that the result is completely independent of τ_1. This is actually to be expected. The τ_1 dependence of correlation functions is determined by the anomalous R-symmetry. Thus any correlator of the σ_a is given by $q_1^m f(q)$ for some m, with f independent of q_1. The singularities of such an object cannot depend upon q_1.

The discussion in this subsection has until now neglected one subtlety. We have computed the singular locus as the locus at which (3.44) have solutions. These equations were obtained by integrating out all of the chiral fields charged under any of the symmetries involved. Of course, there can be intermediate situations in which some factors of G are Higgsed by massless chiral fields and others are in the Coulomb phase. (This simply generalizes the situation in the previous paragraph in which g_1 was Higgsed.) In this case one should integrate out only the massive chiral fields, and seek solutions to the equations of motion for the massless twisted chiral σ_a fields. This will lead to extra components of the singular locus. Indeed the divisor computed above will reproduce only some of the asymptotic singularities predicted in the semiclassical approximation. The extra components will interpolate smoothly between the remaining limits. To extract a coherent picture from this idea we need to understand which limits are joined, or in other words which subsets of the chiral fields can be integrated out to yield a consistent low-energy theory for the remaining light fields. The structure we are looking for is the following. Consider a linear subspace h of $S = \{s \in \mathbb{R}^{n-d} | \sum_{a=1}^{n-d} s_a \sum_{i=1}^{n} Q_i^a = 0\} \sim \mathbb{R}^{n-d-1}$, of dimension k, and let $H \subset G$ be the corresponding subgroup (of rank k). We divide the chiral fields into two subsets, the set $\{\phi_i\}_{i\in I}$ of fields charged under some element of H and its complement, the set $\{\phi_i\}_{i\in I^c}$ of H-invariant fields. We then ask whether a component of the singular locus exists such that at a generic point in it the σ field is in a generic direction within the subspace h. This means the chiral fields ϕ_i are massless for $i \in I^c$ and that their expectation values will lead to a mass matrix for σ whose zero eigenspace is h. We then compute a twisted superpotential $\widetilde{W}$ for the massless components (in h) of σ. We can choose coordinates such that these massless components are $\sigma_{n-d-k+1}, \ldots, \sigma_{n-d}$, and write

$$\widetilde{W}(\Sigma) = \frac{1}{2\sqrt{2}} \sum_{a=n-d-k+1}^{n-d} \Sigma_a \left(i\hat{\tau}_a - \frac{1}{2\pi} \sum_{i\in I} Q_i^a \log(\sqrt{2} \sum_{b=n-d-k+1}^{n-d} Q_i^b \Sigma_b/\Lambda) \right) . \tag{3.46}$$

As above we will then find solutions to the equations of motion derived from this for τ in a codimension-one subvariety, of the form

$$\prod_{i\in I}\left(\sum_{b=n-d-k+1}^{n-d} Q_i^b\sigma_b\right)^{Q_i^a} = e^{2\pi i\tau_a}\ , a=n{-}d{-}k{+}1,\ldots,n{-}d\ . \tag{3.47}$$

When will this construction lead to a component of the singular locus? The special status of the origin in r-space as the unique point at which all of the gauge symmetries can be unbroken has been removed (there is in fact no such point). Instead, along the singular divisor, σ rotates smoothly from one asymptotic direction to another. Thus we expect the components of the true singular locus not to end in the strong coupling region (as those corresponding to convex cones do classically), but to have as boundaries only the semiclassical limits. To check this we consider the classical limit and compute the D-terms using only the massless fields $\{\phi_i\}_{i\in I^c}$. Setting these to zero will restrict r to a cone, which is precisely the cone spanned by the vectors $\{Q_i\}_{i\in I^c}$. If this cone is in fact all of $\mathbb{R}^{n-d-k-1}$, then the singular locus computed above can extend to the classical boundaries of τ-space and there is a component of the singular locus given by the above data. Otherwise, we will find that h is contained in a larger subspace (possibly all of S) and that in the strong coupling regime the singular locus interpolates between the asymptotic limits we have used and others so that in the interior σ rotates out of h. This condition can be naturally expressed in terms of the combinatorial presentation of V. Given a subset $I\subset\{1,\ldots,n\}$ as above, the condition on Q_i is precisely the condition that the set $\{v_i\}_{i\in I}$ is the set of lattice points in a face of the polyhedron $\mathcal{P}$, the convex hull of the v_i. This construction of the singular locus which contains an irreducible component for each face of $\mathcal{P}$ is familiar from studies of **B** model moduli spaces, as we will discuss in detail in section five.

Example 1.

In this example the discussion is completely trivial, since there *is* no singular locus.

Example 2.

The one true singularity in this model, at $\tau_2 = i\frac{\log 2}{\pi}$, is once more a rather trivial example of the above with H generated by g_2. We will see a better example of these phenomena in the models of section four.

We have already managed to extract some exact results from our one-loop computation of (3.36). But we can do more; the correlation functions of the twisted chiral fields σ_a are holomorphic in τ (and do not depend on the location in the worldsheet at which the operators are inserted). We can use (3.36) to obtain exact relations among these which in some cases suffice to compute them all exactly.[14] In fact, for values of r outside the cone $\mathcal{K}_c$ (recall such values always exist for compact V) the theory is far simpler to analyze. The massless fields are just σ_a and these are governed by the interaction (3.36). The correlation functions are then simply products of the expectation values for σ_a derived from (3.44). Note that here we do *not* drop the first equation (indeed there will be solutions at large σ_1 as in (3.42)); the equations are then not homogeneous and determine isolated vacua for generic τ_a in the complement of $\mathcal{K}_c$. The equations (3.44) are then interpreted as constraints on the correlation functions. Since the latter are holomorphic in τ, the relations must continue to hold when we analytically continue to other regions in parameter space. Of course the σ_a should then be considered as operators and not as numbers. Since some of the Q_i^a are negative, this equation may not make sense as written; however, we can take combinations of (3.44) such that the powers on the left-hand side are always positive. These then give a set of nonlinear relations on the σ_a, as we discuss in the next subsection. In some cases these are sufficient to determine the correlators completely.

3.6. The Topological Model

The above discussion has been somewhat imprecise. In particular, the description we have given is a semiclassical one relevant at small e. We have used this reasoning to understand the low-energy (large-e) behavior. Ultimately, the justification for this is that we will limit ourselves here to properties of the supersymmetric theory which are in fact *independent* of e. These quasitopological properties are shared by the topological field theory obtained by "twisting". We now turn to discuss this theory. We will initially base our discussion in the Kähler cone; later we shall see how computations can be done in other phases.

Like any $N = 2$ supersymmetric theory, the model described in the previous subsections can be twisted to obtain a topological field theory [3]. We will study the **A** twist. In this model, the supercharges Q_- and $\overline{Q}_+$ become worldsheet scalars. These generate a $(0|2)$ dimensional supergroup $\mathcal{F}$ of symmetries. The spin-$\frac{1}{2}$ fermions λ, $\psi^{\pm}$ and their

[14] We thank N. Seiberg and E. Witten for urging us to solve the model in this fashion.

conjugates become either one-forms on Σ or scalars. Setting $Q = Q_- + \overline{Q}_+$ we have $Q^2 = 0$. If we restrict attention to correlation functions of operators that are Q-closed, then all Q-exact operators decouple (vanish in correlators). We can thus consider a theory in which the operators are cohomology classes of Q. Correlation functions in this theory are independent of the worldsheet metric because the twisted energy-momentum stress tensor, which couples to the metric, is itself Q-exact. In particular, all correlators are scale-invariant (and the gauge kinetic term, coupling to e, is also exact). Thus we can compute directly in the weak-coupling (high energy) limit quantities which will be relevant to the strongly-coupled theory in the infrared.

The twisting procedure will change the anomaly equation (3.26). By modifying the spins of the fermions we have added to the gauge anomaly (3.26) a gravitational anomaly from the coupling of the fermions to the spin connection on Σ. The modified equation is now

$$\Delta(Q_R) = -\Delta(Q_L) = \frac{d}{2}\chi(\Sigma) - \frac{1}{2\pi}\sum_{i=1}^{n}\left(\sum Q_i^a \int_\Sigma d^2z\, f_a\right) \,, \tag{3.48}$$

the additive constant being simply the difference between the number of left- and right-moving fermions to which the current couples. The difference $Q_L - Q_R$ is called ghost number.

In the nonlinear sigma model the cohomology of Q is precisely the de Rham cohomology $H^*_{\rm DR}(V)$, discussed in subsection *3.1*. By the arguments of the previous paragraph we expect to find a similar structure for the local observables in the linear model. The Q variations of the fields are:

$$\begin{aligned}
&[Q, v_z] = -i\lambda_z^+ && \{Q, \lambda_z^+\} = 2\sqrt{2}\partial_z\sigma \\
&[Q, v_{\bar z}] = -i\bar\lambda_{\bar z}^- && \{Q, \lambda^-\} = i(D+f) \\
&[Q, \sigma] = 0 && \{Q, \bar\lambda^+\} = i(D+f) \\
&[Q, \bar\sigma] = i\sqrt{2}(\bar\lambda^+ - \lambda^-) && \{Q, \bar\lambda_{\bar z}^-\} = 2\sqrt{2}\partial_{\bar z}\sigma
\end{aligned} \tag{3.49}$$

for each of the gauge multiplets, and

$$\begin{aligned}
& && \{Q, \chi^i\} = 2\sum_a Q_i^a \bar\sigma_a \phi^i \\
&[Q, \phi^i] = \sqrt{2}\chi^i && \{Q, \bar\chi^{\bar\imath}\} = -2\sum_a Q_i^a \bar\sigma_a \bar\phi^{\bar\imath} \\
&[Q, \bar\phi^{\bar\imath}] = -\sqrt{2}\bar\chi^{\bar\imath} && \{Q, \psi_{\bar z}^i\} = 2\sqrt{2} i D_{\bar z}\phi^i + \sqrt{2}F_{\bar z}^i \\
& && \{Q, \bar\psi_z^{\bar\imath}\} = 2\sqrt{2} i D_z \bar\phi^{\bar\imath} + \sqrt{2}\overline{F}_z^{\bar\imath}
\end{aligned} \tag{3.50}$$

for the chiral multiplets.

The supersymmetry variations above lead us to identify representatives for the cohomology of Q in the space of local operators: every class can be represented by a function of the σ_a's. That is, the Q-variations realize the Cartan model for $\mathcal{G}$-equivariant cohomology of field space, where $\mathcal{G}$ is the group of gauge transformations. A recent review of this procedure can be found in [55]. The cohomology is graded by ghost number, with Q carrying ghost number one; σ_a carries ghost number two. Note that the nonlinear relations (3.10) are not satisfied by the operators. We will identify the correct nonlinear relations using the correlation functions as discussed in section two. We wish to compute correlation functions in the low-energy model, and hence look for the map between the operators in the nonlinear sigma model and the space obtained in the GLSM. Restricting to the zero modes, the GLSM operators will be given by G-equivariant cohomology. This in turn is naturally identified with the cohomology of V, leading to the identification $\sigma_a \sim \eta_a$ to within a normalization.

In a topological field theory, given a local Q-closed operator we can form other Q-closed operators by the descent equations. These lead to Q-cohomology classes expressed as the integrals over cycles on the worldsheet of appropriate forms. The two-form operators thus obtained yield deformations of the action that preserve the topological invariance. Applying this method to σ_a leads to $\int_\Sigma (D_a - f_a)$, the deformation coupling to τ. This is in line with the identification made above, of course. Further, one can show that the operator coupling to $\bar{\tau}$ is Q-exact. The correlation functions of σ_a will thus be holomorphic functions of τ.

The nonlinear relations among the σ_a will be determined by the correlation functions we compute in the sequel. However, a subset of these relations can be computed directly from the twisted superpotential (3.36), as mentioned in the previous subsection. If for each i we fix an operator δ_i which in Q-cohomology is given by

$$\delta_i \sim \frac{\sqrt{2}e}{\Lambda} \sum_{a=1}^{n-d} Q_i^a \sigma_a \ , \tag{3.51}$$

then we can write the relations (3.44) in the form

$$\prod_{i=1}^{n} \delta_i^{Q_i^a} = e^{2\pi i \tau_a} \ . \tag{3.52}$$

Relations of this type had already been proposed by Batyrev [8], in the form

$$\prod_{i=1}^{n} \xi_i^{Q_i^a} = e^{2\pi i t_a} \ . \tag{3.53}$$

In fact, for smooth V Batyrev showed that such relations would suffice to compute the quantum cohomology of V (see a fuller discussion in subsection *3.9*). We will recover these relations by a direct study of the moduli spaces of instantons (refining the analysis of Batyrev) in the sequel. Our direct approach has the advantage of not being limited to smooth V, and will prove very useful in the next sections. At this point, we can use (3.52) and (3.53) to find the normalization constant in our identification of operators with divisors and to verify that indeed (3.32) is not corrected. Notice that for values of a satisfying (3.39) the constants in (3.51) drop out of (3.52) (in particular, Λ does not appear as expected since the instantons in this factor do not contribute to the anomaly).

3.7. Reduction to Moduli Space

The computation of correlation functions in the topological field theory is greatly simplified by the existence of the $(0|2)$ dimensional symmetry group $\mathcal{F}$. The path integral computing correlation functions of Q-closed operators (for which $\mathcal{F}$-invariant representatives can be chosen) reduces to an integral over the fixed point set of $\mathcal{F}$. Fluctuations about this can be treated exactly in the Gaussian approximation. Furthermore, exact cancellation between the bosonic and fermionic determinants means they can be altogether dropped. (Note that in general the ratio of determinants can be ± 1; in the present case it is in fact just 1, as we discuss more fully in section four.) In the nonlinear sigma model, the fixed point set of $\mathcal{F}$ is the space of holomorphic maps $\Sigma \to V$ and its irreducible components are classified by the degree of the map. The contribution of each component can be expressed as an intersection computation in that space, but the moduli spaces are highly nontrivial and noncompact and explicit computations are extremely difficult. We will find a qualitatively similar picture in the linear model. The essential difference will be that the moduli spaces will be simple and the computation tractable.

The fixed point set of $\mathcal{F}$ is found by considering field configurations annihilated by $\overline{Q}_+$ and Q_-. As in [6] this leads to

$$d\sigma_a = 0 \tag{3.54a}$$

$$\sum_{a=1}^{n-d} Q_i^a \sigma_a \phi_i = 0 \tag{3.54b}$$

$$D_a + f_a = 0 \tag{3.54c}$$

$$D_{\bar{z}} \phi_i = 0 \ , \tag{3.54d}$$

where $D_{\bar{z}}$ is a covariant derivative constructed from the gauge connection. The space of solutions to these will split into "instanton sectors" labeled by

$$n_a = -\frac{1}{2\pi} \int d^2 z \, f_a \ . \tag{3.55}$$

The action for a solution of (3.54) is

$$L = -2\pi i \sum_{a=1}^{n-d} \tau_a n_a \ . \tag{3.56}$$

For r in the interior of the cone, (3.54a–b) lead to $\sigma = 0$, leaving (3.54c–d) to determine ϕ and f. The equations are of course gauge-invariant; the moduli space $\mathcal{M}_{\vec{n}}$ of solutions at instanton number $\vec{n}$ is a quotient of the space of solutions satisfying (3.55) by the group of gauge transformations

$$\phi_i \to e^{i\Sigma_a Q_i^a \epsilon_a} \phi_i \ , \qquad v_a \to v_a - d\epsilon_a \ . \tag{3.57}$$

What makes the computation tractable is the fact that this quotient can be recast as a toric variety. The first equation (3.54c) is in fact invariant under (3.57) with ϵ a *complex* function on Σ, while the second (with D expressed in terms of ϕ using (3.27)) is not. The essential observation [6,56,57] is that for each solution of (3.54c) there is at most one value of $|\epsilon|$ transforming it to a solution of (3.54d) as well. Thus $\mathcal{M}_{\vec{n}}$ can be expressed as the set of solutions of (3.54c) which can be transformed this way modulo complex gauge transformations.

By an appropriate complex gauge transformation we can make v a holomorphic connection. Then (3.54c) states simply that ϕ_i is a holomorphic section of a line bundle over Σ of degree $d_i(\vec{n}) = \sum_a Q_i^a n_a$. Restricting attention to Σ of genus zero, this is the line bundle $\mathcal{O}(d_i)$. For $d_i < 0$ such sections do not exist and we have $\phi_i = 0$. For $d_i \geq 0$ there is a $(d_i + 1)$-dimensional vector space of sections. These can be expressed as homogeneous polynomials of degree d_i in the homogeneous coordinates (s, t) on Σ

$$\phi_i = \phi_{i0} s^{d_i} + \phi_{i1} s^{d_i - 1} t + \cdots + \phi_{id_i} t^{d_i} \ , \tag{3.58}$$

i.e., we can think of $\{\phi_i\}$ as a $G_{\mathbb{C}}$-equivariant map $\mathbb{C}^2 \to Y$. It is now easy to see which sets of sections do not lead to a solution of (3.54d) for any $|\epsilon|$. These are those solutions for which the image lies completely in F, determined by r as in subsection *3.1*. We denote this set of solutions $F_{\vec{n}}$. We have not completely fixed the complex gauge invariance.

Complex gauge transformations with constant ϵ will not affect our choice of connection, so we must still quotient our space by the action of these. In summary, we have the following expression for the moduli spaces (compare [58]):

$$\mathcal{M}_{\vec{n}} = (Y_{\vec{n}} - F_{\vec{n}})/T \tag{3.59}$$

where $Y_{\vec{n}} = \bigoplus_{i=1}^{n} H^0(\mathcal{O}(d_i))$ and $T = G_{\mathbb{C}}$ acts by $\phi_{ij} \to (\prod_a \lambda_a^{Q_i^a})\phi_{ij}$. The components ϕ_{ij} of the ϕ_i's give coordinates on $Y_{\vec{n}}$. Since ϕ_i has $\max\{0, d_i+1\}$ components, we find

$$\dim Y_{\vec{n}} = \sum_{i:d_i \geq 0} (d_i + 1) \tag{3.60}$$

and hence

$$\begin{aligned} \dim \mathcal{M}_{\vec{n}} = \dim Y_{\vec{n}} - (n-d) = \\ \left(d + \sum_{i=1}^{n} d_i \right) + \sum_{i:d_i \leq -1} (-d_i - 1) \geq d + \sum_{i=1}^{n} d_i \ . \end{aligned} \tag{3.61}$$

A similar analysis can be carried out for worldsheets Σ of higher genus: each ϕ_i is a section of a line bundle $\mathcal{L}_i$ on Σ, and these line bundles must have compatible degrees. (In fact, a bit more compatibility is required: there must be isomorphisms among certain tensor powers of these bundles – see [58] for details of this construction.)

The toric variety $\mathcal{M}_{\vec{n}}$ can also be described via symplectic reduction. If we start with a moment map D (depending on a choice of r in $\mathcal{K}_q$) which defines V in the form $D^{-1}(0)/G$, then F is characterized as the set of points in Y whose $G_{\mathbb{C}}$-orbit does not meet $D^{-1}(0)$. If we replace each term $|\phi_i|^2$ in D with a corresponding term $\sum_{j=0}^{d_i} |\phi_{ij}|^2$ to form a map $D_{\vec{n}}$

$$D_{\vec{n},a} = \sum_{i=1}^{n} Q_i^a \left(\sum_{j=1}^{d_i} |\phi_{ij}|^2 \right) - r_a \tag{3.62}$$

then we see that $D_{\vec{n}}$ serves as a moment map in this context with $F_{\vec{n}}$ being precisely the set of points in $Y_{\vec{n}}$ whose $G_{\mathbb{C}}$-orbit does not meet $D_{\vec{n}}^{-1}(0)$. Note that the constants r_a which determine the moment map are the same as those used for the moment map of V itself.

We observe that the moduli space $\mathcal{M}_{\vec{n}}$ must be empty unless $\vec{n}$ belongs to the dual of the cone of allowed r's for this model. Let $\vec{n}$ be such that $\sum n_a r_a < 0$. We consider $\frac{1}{e^2} \sum n_a D_a$, which takes the form

$$\sum_i (\sum_a n_a Q_i^a)|x_i|^2 - (\sum n_a r_a). \tag{3.63}$$

From this form, we see that if $I = \{i \mid \sum_a n_a Q_i^a < 0\}$, then F contains the set $\{x_i = 0 \quad \forall i \in I\}$. But now for polynomials ϕ_i in this sector, $d_i < 0$ for $i \in I$ which means that all such ϕ_i $(i \in I)$ must vanish identically. Thus, the corresponding collection (ϕ_i) lies in $F_{\vec{n}}$, i.e., $Y_{\vec{n}} = F_{\vec{n}}$ so $\mathcal{M}_{\vec{n}}$ is empty. Since this holds for any r which determines the same toric variety, the moduli spaces must be empty for any $\vec{n}$ not in the (closed) cone $\mathcal{K}^\vee$ dual to the cone $\mathcal{K}$ in r-space determining the phase in which we compute.

The cohomology of the moduli space $\mathcal{M}_{\vec{n}}$ is generated by classes of toric divisors. The coordinates in the space $Y_{\vec{n}}$ are given by the set of coefficients ϕ_{ij} of the sections ϕ_i. Each can be set to zero, giving divisor classes ξ_{ij}. Among the linear relations we find that for a fixed i, all ξ_{ij}'s are linearly equivalent to each other. (This essentially says that we can vary the insertion point of the operator without affecting the cohomology class on the moduli space and hence with no effect on correlation functions as expected.) Let us identify all of these with a fixed class ξ_i (more precisely, $(\xi_i)_{\vec{n}}$).

These classes can be written in terms of some classes $(\eta_a)_{\vec{n}}$ as follows:

$$(\xi_{ij})_{\vec{n}} = (\xi_i)_{\vec{n}} = \sum_{a=1}^{n-d} Q_i^a (\eta_a)_{\vec{n}} \ . \tag{3.64}$$

The $(\eta_a)_{\vec{n}}$'s generate the cohomology in this case.

It is useful at this point to compare the rather simple instanton moduli spaces (3.59) to the intractable ones found in the nonlinear sigma model. In fact, the equivariant maps ϕ_i are closely related to maps $\Sigma \to V$. In particular, setting $\vec{n} = 0$ we immediately recover $\mathcal{M}_0 = V$. However, not every instanton represents an actual map from the worldsheet to V. A collection ϕ_i which sends *any* point from $\mathbb{C}^2 - \{(0,0)\}$ to F is disqualified from being a true map. Following Witten, we call these the *pointlike instantons*, and note that they comprise a subset of $\mathcal{M}_{\vec{n}}$ of positive codimension. The moduli space $\mathcal{M}_{\vec{n}}$ is obtained from the corresponding moduli space in the nonlinear model by adding these configurations. We can interpret them physically by returning to (3.54c). For points in $\mathcal{M}_{\vec{n}}$ which correspond to true maps we see that the curvature f is small. For pointlike solutions, on the other hand, there are points on the worldsheet at which the curvature is large (of order r). The strong fluctuations of the gauge field around these points extend to a distance of order the Compton wavelength of the massive gauge boson. From the point of view of the low-energy theory these field configurations are singular at a point on Σ. Mathematically, the pointlike instantons lead to a natural compactification of the space of holomorphic maps from $\mathbb{P}^1$

to V. It is the natural occurrence of this compactification, and the extreme simplicity of the compactified space, that make the instanton computation tractable.

This comparison of the instanton spaces will also allow us to show that the relation (3.32) between the GLSM parameters and those of the low-energy nonlinear model is exact. Since the instanton spaces for the GLSM differ from those for the nonlinear model only in positive codimension, and since in either case the contributions to correlation functions arise by intersection calculations, or equivalently by integrating a closed form of top degree, the contribution of a given instanton sector is expected to be the same in both theories. Thus, for the models studied here, the relation $t_a = \tau_a$ holds *exactly*, and the computations of the next subsection yield the Gromov–Witten invariants of V directly. (The analogous statement for the linear models related to Calabi–Yau hypersurfaces $M \subset V$ is false, as we shall see later.)

Example 1.

In the first example, working in the Kähler cone $r > 0$, we find $\mathcal{K}^\vee$ given by $n \geq 0$. Thus ϕ_i is a section of $\mathcal{O}(n)$. Pointlike instantons are configurations in which the maps ϕ_i have a common zero. The set F_n contains the zero map, and $Y_n = \mathbb{C}^{5(n+1)}$. Thus $\mathcal{M}_n = (Y_n - F_n)/\mathbb{C}^* = \mathbb{P}^{5n+4}$. This is a compactification of the space of degree n maps to $V = \mathbb{P}^4$. Note that $\dim_{\mathbb{C}} \mathcal{M}_n = 4 + 5n$ as expected from (3.61). The cohomology of $\mathcal{M}_n$ is generated by the hyperplane class $\xi_{ij} = \eta \quad \forall i, j$ subject to the relation (analog of (3.10)) $\eta^{5n+5} = 0$.

Example 2.

In the second example, working once more in the Kähler cone $\mathcal{K}_V = \{r_1, r_2 > 0\}$, we have $\mathcal{K}^\vee = \{n_1, n_2 \geq 0\}$ and $d = (n_2, n_2, n_1, n_1, n_1, n_1 - 2n_2)$. Here there are two cases to consider. The first is $n_1 - 2n_2 \geq 0$. In this case we find $Y_{\vec{n}} = \mathbb{C}^{4n_1+6}$. Comparing d and (3.4) we find the nonlinear relations defining $F_{\vec{n}}$ are

$$\begin{aligned} \xi_1^{n_2+1}\xi_2^{n_2+1} &= \eta_2^{2n_2+2} = 0 \\ \xi_3^{n_1+1}\xi_4^{n_1+1}\xi_5^{n_1+1}\xi_6^{n_1-2n_2+1} &= \eta_1^{3(n_1+1)}(\eta_1 - 2\eta_2)^{n_1-2n_2+1} = 0 \; . \end{aligned} \tag{3.65}$$

So $\mathcal{M}_{\vec{n}} = (Y_{\vec{n}} - F_{\vec{n}})/{\mathbb{C}^*}^2$ is defined as a toric variety of dimension $4n_1 + 4$. On the other hand, if $n_1 - 2n_2 < 0$ we see that d_6 is negative, hence $\phi_6 \equiv 0$. When this obtains the two $U(1)$'s in $G = U(1)^2$ act on disjoint sets of fields, hence $\mathcal{M}_{\vec{n}} = \mathbb{P}^{2n_2+1} \times \mathbb{P}^{3n_1+2}$. The hyperplane class of the first factor is η_2, and of the second η_1. The total dimension is $3n_1 + 2n_2 + 3$.

3.8. Computing Correlators

We are now almost ready to compute the correlation functions

$$Y_{a_1 \ldots a_s} = \langle \sigma_{a_1}(z_1) \cdots \sigma_{a_s}(z_s) \rangle \tag{3.66}$$

where $z_i \in \Sigma$ and of course the result does not depend upon the choice of the z_i's. We compute Y in an instanton expansion as

$$Y_{a_1 \ldots a_s} = \sum_{\vec{n} \in \mathcal{K}^\vee} Y^{\vec{n}}_{a_1 \ldots a_s} \prod_{a=1}^{n-d} q_a^{n_a} \ , \tag{3.67}$$

where $q_a = e^{2\pi i \tau_a}$. (This series is expected to converge for $|q_a|$ sufficiently small.) The contributions $Y^{\vec{n}}_{a_1 \ldots a_s}$ are given by the restriction of the path integral to this component of the moduli space. Clearly, this will vanish unless the anomalous ghost number conservation law (3.48) is satisfied, i.e., $s = d + \sum_{i=1}^n d_i$. This is in complete accord with the discussion in subsection *3.4*. The sum (3.67) can diverge at small r_a only if it is an infinite series in q_a; this is turn can only occur if (3.39) is satisfied.

The $Y^{\vec{n}}_{a_1 \ldots a_s}$ will be given by an intersection computation on $\mathcal{M}_{\vec{n}}$. To each insertion of σ_a we associate a class in $H^*(\mathcal{M}_{\vec{n}})$. This can be determined using the methods of [55]. In our context, we can make the identification more directly, in a similar manner to our treatment of the zero modes. In this case, restricting attention to field configurations of the form (3.58) we see that (3.50) leads to G-equivariant cohomology of $Y_{\vec{n}}$. To within a normalization, the class corresponding to σ_a is thus η_a. We will assume this holds exactly (the normalization factor is independent of $\vec{n}$), though we do not have a complete justification for this choice. We can consider a change of this normalization as a change of the contact terms between the inserted class and the perturbing operator, so our choice is simply a choice of some natural contact terms.[15] We note that this normalization is suggested by the correspondence between (3.52) and (3.53). Making this identification we find, for the case $d_i \geq 0$ that

$$Y^{\vec{n}}_{a_1 \ldots a_s} = \langle (\eta_{a_1})_{\vec{n}} \cdot (\eta_{a_2})_{\vec{n}} \cdots \cdot (\eta_{a_s})_{\vec{n}} \rangle_{\vec{n}} \tag{3.68}$$

when (3.48) is satisfied, zero otherwise. Here $\langle \ \rangle_{\vec{n}}$ denotes the intersection form on $\mathcal{M}_{\vec{n}}$, and when no confusion is likely we will simply write η_a, the lift to moduli space being

[15] E. Witten has suggested to us that the computation of quantum cohomology could be carried out by the methods of ref. [59], which would also determine the normalization.

understood. We have ignored the constant normalization factors in (3.68) and will continue to do so.

When some $d_i < 0$ the dimension of moduli space is too large. This means that the section whose zero set is $\mathcal{M}_{\vec{n}}$ is not generic. In this case the solution is well-known [10]. The contribution is obtained by inserting in (3.68) the Euler class $\chi_{\vec{n}}$ of the obstruction bundle. Physically, in this case there are also $\psi^i_{\bar{z}}$ zero modes; $\chi_{\vec{n}}$ is obtained by integrating over these. Since the obstruction bundle has a G-action, it is determined by the corresponding representation of G. It follows that, up to a normalization, $\chi_{\vec{n}}$ takes the form

$$\chi_{\vec{n}} = \prod_{d_i<0} (\xi_i)_{\vec{n}}^{-d_i-1} \ , \tag{3.69}$$

where the product is an intersection as above, and ξ_i is defined by (3.64). We assume that (3.69) in fact gives the correct normalization. It may appear strange to insert ξ_i, since the maps take values in the subset $\phi_i = 0$ to which this is certainly not transverse. The point is that we use (3.64) to define the insertion, and the η_a do have representatives meeting $\mathcal{M}_{\vec{n}}$ transversely. The general formula is thus

$$Y^{\vec{n}}_{a_1 \ldots a_s} = \langle (\eta_{a_1})_{\vec{n}} \cdot (\eta_{a_2})_{\vec{n}} \cdot \cdots (\eta_{a_s})_{\vec{n}} \chi_{\vec{n}} \rangle_{\vec{n}} \ . \tag{3.70}$$

Example 1.

Since the index a takes but one value in this example, we will simplify notation by dropping it and introducing Y_s to stand for the previous $Y_{a \ldots a}$ (s factors of a). Here χ_n is trivial and we find $Y^n_s = \delta_{s,5n+4}$, so that

$$Y_{5m+4} = \sum_{n \geq 0} q^n Y^n_{5m+4} = q^m \ , \tag{3.71}$$

all others vanishing. All correlation functions are analytic in q and there is no singularity, as predicted in subsection *3.4* above.

Example 2.

In the second example we will have to work a little harder. The correlation functions we compute will be $Y_{m_1,m_2} = \langle \sigma_1^{m_1} \sigma_2^{m_2} \rangle$. Ghost number conservation implies $Y^{\vec{n}}_{m_1,m_2} = 0$ unless $m_1 + m_2 = 4n_1 + 4$. We thus need $Y^{\vec{n}}_{4n_1+4-m_2,m_2}$, and to get it we will need to study the intersection form on $\mathcal{M}_{\vec{n}}$. Let us start with the case $n_1 - 2n_2 \geq 0$. In this case we will use the relations in the algebra to determine the expectation function to within

a normalization as discussed in section two. To determine this we need to compute one intersection and toric methods yield easily

$$\langle \eta_2^{2n_2+1} \eta_1^{4n_1-2n_2+3} \rangle_{\vec{n}} = 1 \ , \tag{3.72}$$

(cf. (3.11)). Using the relations we can reduce any nonzero intersection to these. Let

$$\varphi_a = 2^{-a} \langle \eta_2^{2n_2+1-a} \eta_1^{4n_1-2n_2+3-a} \rangle_{\vec{n}} \tag{3.73}$$

for $0 \le a \le 2n_2+1$, and set $\varphi_a = 0$ for $a < 0$. Then $\varphi_0 = 1$ is the normalization condition, and we have the following recursion relation by expanding the second of (3.65)

$$\varphi_a + \sum_{j=1}^{n_1-2n_2+1} (-1)^j \binom{n_1 - 2n_2 + 1}{j} \varphi_{a-j} = 0 \tag{3.74}$$

for $1 \le a \le 2n_2 + 1$ determining φ_a. This is solved by

$$\varphi_a = \binom{n_1 - 2n_2 + a}{a} \ , \tag{3.75}$$

(see appendix A) which yields finally

$$Y^{\vec{n}}_{4n_1+4-m_2,m_2} = 2^{2n_2+1-m_2} \binom{n_1 + 1 - m_2}{2n_2 + 1 - m_2} \ . \tag{3.76}$$

Let us now consider the case $n_1 - 2n_2 < 0$. The moduli space was described above. Since $d_6 < 0$ the contribution is

$$Y^{\vec{n}}_{4n_1+4-m_2,m_2} = \langle \eta_2^{m_2} \eta_1^{4n_1+4-m_2} \chi \rangle_{\vec{n}} \ , \tag{3.77}$$

where $\chi = \xi_6^{2n_2-n_1-1} = (\eta_1 - 2\eta_2)^{2n_2-n_1-1}$. The only nonzero contribution to (3.77) is simply the coefficient of $\eta_1^{3n_1+2}\eta_2^{2n_2+1}$, hence expanding χ

$$Y^{\vec{n}}_{4n_1+4-m_2,m_2} = (-2)^{2n_2+1-m_2} \binom{2n_2 - n_1 - 1}{2n_2 + 1 - m_2} = 2^{2n_2+1-m_2} \binom{n_1 + 1 - m_2}{2n_2 + 1 - m_2} \ . \tag{3.78}$$

Remarkably, this is identical to (3.76). We can now sum the instanton series to obtain

$$Y_{4n_1+4-m_2,m_2} = q_1^{n_1} \sum_{n_2 \ge 0} 2^{2n_2+1-m_2} \binom{n_1 + 1 - m_2}{2n_2 + 1 - m_2} q_2^{n_2} \ . \tag{3.79}$$

One observes that for $m_2 > n_1 + 1$ the series is infinite, and its sum has a pole at $q_2 = 1/4$, in complete agreement with the discussion of subsection *3.5* above.

3.9. Quantum Cohomology

In the preceding subsection we have completely solved the model and computed all the correlators. As observed in section two, these determine a deformation of the cohomology algebra of V to an object known as the *quantum cohomology algebra*. This is essentially the ring of local operators in the topological field theory. This ring can be viewed as a deformation (with parameters q_a) of the cohomology ring $H^*(V)$, which approximates the latter in the limit $q_a \to 0$, i.e. as we move in r-space to a point deep in the interior of the Kähler cone. Conversely, the structure of the algebra determines the correlation functions to within an overall (possibly q-dependent) normalization. In subsection *3.6* we computed some of the relations in this algebra (3.44). In this subsection we will show how the direct analysis reproduces these relations. Our treatment closely follows the work of [8]; the method of proof will be useful in what follows. If V is smooth, these relations determine the algebra completely as we discuss. We stress however, that the computations given above are valid without this assumption. In fact, in appendix B we consider a model in which V is not smooth and show that the details of the solution for such a model differ somewhat from the smooth case.

The correlation functions (3.66), since they do not depend on the points z_i at which we insert the local operators, can be interpreted as defining a linear function, the "expectation function," on $\mathcal{Y} = \mathbb{C}[\sigma_1, \ldots, \sigma_{n-d}]$, the ring of polynomials in these formal variables. This, as pointed out earlier, is essentially the ring of local observables in the theory. This statement is somewhat misleading, however. There is an ideal $\mathcal{J}$ in $\mathcal{Y}$ annihilated by all correlation functions, i.e. for $\mathcal{P} \in \mathcal{J}$ we have $\langle \mathcal{O}\mathcal{P} \rangle = 0$ for all $\mathcal{O} \in \mathcal{Y}$. The ideal $\mathcal{J}$ of course depends on q. The true space of local operators is the quotient space $\mathcal{Y}/\mathcal{J}(q)$. The problem of computing the quantum cohomology of V is thus equivalent to computing the ideal $\mathcal{J}(q)$.[16] It will prove convenient, in what follows, to work instead with the presentation $\mathcal{Y} = \mathbb{C}[\delta_1, \ldots, \delta_n]/\mathcal{L}$ where the δ are determined by (3.51) and $\mathcal{L}$ is the ideal generated by the linear relations among these following from (3.51). (Note that just as σ_a induces cohomology classes $(\eta_a)_{\vec{n}}$ in the moduli spaces, δ_i induces the corresponding cohomology classes $(\xi_i)_{\vec{n}}$ given by (3.64).)

[16] This approach to studying quantum cohomology by finding generators σ and analyzing the ideal $\mathcal{J}$ of relations among them (as determined by the correlation functions) has been systematized by Siebert and Tian [61].

As an example, we can set $q_a = 0$. In this limit we expect to reproduce the classical cohomology ring of V. Physically, the contributions of nontrivial instanton sectors are suppressed in this limit and we obtain correlators as intersection computations on $\mathcal{M}_0 = V$ as observed above. The ideal $\mathcal{J}(0)$ is then simply related to the set F of excluded intersections of hypersurfaces. Consider a component of F given by $\phi_{i_1} = \cdots = \phi_{i_p} = 0$ (where $\phi_{i_1}, \ldots, \phi_{i_p}$ is a "primitive collection" as before). Then of course any correlation function $\langle \mathcal{O}\delta_{i_1} \cdots \delta_{i_p}\rangle_0 = \langle \mathcal{O}\xi_{i_1} \cdots \xi_{i_p}\rangle$ vanishes in this limit because the corresponding hypersurfaces do not meet in V. This is precisely the presentation of the cohomology ring of V which appeared above in subsection *3.1*.

Let us now move away from the locus $q_a = 0$. When $q_a \neq 0$, there can be nonzero contributions to correlation functions containing operators from $\mathcal{J}(0)$. In the nonlinear language, these come from nontrivial rational curves in V which intersect the p divisors, even though these do not intersect in V. We can readily study these by taking advantage of the fact that correlators in the topological field theory do not depend on z_i. Essentially we will use this freedom to insert the operators at the *same* point on Σ. As always, the local operator corresponding to this situation suffers from ambiguities related to contact terms. We will essentially be making the analog of the canonical choice, i.e., we will use the point-splitting definition and use computations with all the operators inserted at a point to learn about the generic situation.

The ideal $\mathcal{J}$ can be constructed as follows (we are here following closely the argument of [8]). We will start by writing one relation for every set $\vec{n}^*$ of $n-d$ integers lying in the $(n-d)$-dimensional cone $\mathcal{K}^+ \subset \mathcal{K}^\vee$ determined by the inequalities $d_i(\vec{n}) \geq 0$. Given such a vector $\vec{n}^*$ we will show that for any operator $\mathcal{O}$ in the ring and any $\vec{n} \in \mathcal{K}^\vee$

$$Y_{\mathcal{O}}^{\vec{n}} = Y_{\mathcal{O}^*}^{\vec{n}+\vec{n}^*} \tag{3.80}$$

where $\mathcal{O}^* = \mathcal{O}\prod_{i=1}^n \delta_i^{d_i^*}$.

To see this, let us first assume that $\vec{n} \in \mathcal{K}^+$. In this case (3.80) is

$$\langle \mathcal{O}\rangle_{\vec{n}} = \langle \mathcal{O}\prod_{i=1}^{n}(\xi_i)_{\vec{n}+\vec{n}^*}^{d_i^*}\rangle_{\vec{n}+\vec{n}^*} \ . \tag{3.81}$$

We can choose explicit representatives for the ξ_i on the right-hand side such that in the space of equivariant maps ϕ_i we impose all of the constraints at $s = 0$ on Σ. This means

that the expression (3.58) for ϕ_i as a homogeneous polynomial of degree $d_i + d_i^*$ takes the form

$$\phi_i = s^{d_i^*} P_i(s,t) \tag{3.82}$$

where P_i is a homogeneous polynomial of degree $d_i \geq 0$. But the space of such polynomials is precisely $Y_{\vec{n}}$. Further, the image lies in F precisely when the maps P_i lie in $F_{\vec{n}}$. Thus choosing these representatives on the right-hand side of (3.81) leads precisely to the left-hand side.

Next, note that if for some i we have $d_i + d_i^* < 0$, the discussion of the previous paragraph is modified as follows. On both sides of (3.80) the moduli space in question is restricted to maps for which $\phi_i = 0$ identically. However, on both sides we must modify (3.81) by inserting the appropriate classes χ. On the left-hand side this contains a factor of $\xi_i^{-d_i-1}$, while on the right-hand side the factor that appears is $\xi_i^{-d_i-d_i^*-1}$. Comparing the two we see that (3.80) holds in this case as well. Finally, we must also address the case $0 < -d_i < d_i^*$. In this case, on the left-hand side of (3.80) we have set $\phi_i = 0$ identically and inserted $\xi_i^{-d_i-1}$. On the right-hand side, we have ϕ_i of degree $d_i + d_i^*$ initially; however, we can use $d_i + d_i^* + 1$ of the explicit insertions of ξ_i to restrict to the set $\phi_i = 0$ identically; this leaves us with $-d_i - 1$ insertions to be imposed in this subset, once more in agreement with (3.80). There is one final case to study, $\vec{n} \notin \mathcal{K}^\vee$ but $\vec{n} + \vec{n}^* \in \mathcal{K}^\vee$. It is easy to see that in this case both sides of (3.80) vanish.

In conclusion, what we have shown is that for each $\vec{n}^* \in \mathcal{K}^+$ we have a relation following from (3.80)

$$\prod_{i=1}^{n} \delta_i^{d_i^*} = \prod_{a=1}^{n-d} q_a^{n_a^*} \ , \tag{3.83}$$

in agreement with (3.52) (the cone $\mathcal{K}^+$ of course contains precisely the combinations leading to non-negative powers of ξ_i referred to there).

If V is smooth these relations generate the ideal $\mathcal{J}(q)$. In fact, the argument to this point has demonstrated only that the ideal generated by our relations is contained in $\mathcal{J}(q)$. The converse, for the case of smooth compact V, essentially follows from the work of Batyrev. Under these hypotheses, he showed that the ring (more precisely algebra) defined by these relations approximates $H^*(V)$ in the limit $q \to 0$. Note that our choice of n^* makes this limit well-defined; thus the dimension of the algebra is independent of q for values away from the singular loci and in particular does not change as $q \to 0$. Now, imposing any additional relations would change the dimension of the ring (at least

for nonzero q, the new relation could be ill-defined at $q = 0$). But this would be in disagreement with the fact that the space of operators at any q is isomorphic as a graded vector space to $H^*(V)$. The restriction to smooth V enters the argument precisely at this point. When V is singular, the relations (3.80) are certainly contained in the ideal $\mathcal{J}$, but they do not suffice to generate it. In appendix B we describe an example of a singular toric variety for which this occurs. Of course, since we solve the model for all values of the parameters, the algebraic solution is valid whenever there is *some* toric model for V which is smooth.

Notice that our choice of contact terms is seen to agree with natural choices made from other points of view for which the nonlinear model was the point of departure. In particular, the τ_a are indeed the canonical coordinates as expected. To construct the canonical basis one begins with the classes η_a. Their products (modulo the relations) are a basis for the algebra; the change of basis to a canonical set is determined by the requirement that all two-point functions be constant.

Example 1.

Here we have seen that $\mathcal{K}^+ = \mathcal{K}^\vee$ is generated by $n = 1$, from which we obtain the relation $\prod_{i=1}^5 \delta_i = \sigma^5 = q$, in agreement with the well-known results for the nonlinear $\mathbb{P}^n$ model [62].

Example 2.

In the second example $\mathcal{K}^+$ is generated by $\vec{n} = (1,0),\ (2,1)$, leading to the relations

$$\begin{aligned} \delta_3\delta_4\delta_5\delta_6 = \sigma_1^3(\sigma_1 - 2\sigma_2) = q_1 \\ \delta_1\delta_2\delta_3^2\delta_4^2\delta_5^2 = \sigma_1^6\sigma_2^2 = q_1^2 q_2\ . \end{aligned} \tag{3.84}$$

Dividing the second of these by the square of the first we obtain

$$\sigma_2^2 = q_2(\sigma_1 - 2\sigma_2)^2\ , \tag{3.85}$$

which together with the first of (3.84) form a deformation of (3.14).

---

4. The Linear Sigma Model for M

Topological sigma models with toric target spaces are interesting as solvable nontrivial field theories. However, the recent interest in toric geometry among physicists has in fact focused on different though related theories. These are the superconformal nonlinear sigma models with Calabi–Yau target spaces that can be embedded as hypersurfaces $M \subset V$ in toric varieties. For these models there exists a conjecture by Batyrev [26] which if true allows one to construct the mirror manifold W as a hypersurface in a toric variety Λ related to V by a combinatorial duality. In [26] the most basic implication of mirror symmetry – the relation between the Hodge numbers of M and W – was verified. In this section we will use the results of our careful study of the model with target space V to study the model with target space $M \subset V$ a Calabi–Yau hypersurface. We will find that the **A** model with target space M is solvable in the same sense (and using the same methods) as that for V. We will return to discuss mirror symmetry and Batyrev's construction in greater detail in section five. The modification to the model of section three which leads to the nonlinear sigma model with target space M was given in [6]. We note that the methods of [6] and probably the extension given here are not restricted to hypersurfaces; however this is the simplest case to study and we restrict ourselves to that case in this paper.

4.1. Toric Geometry on the Other Leg

A toric variety V as described in subsection *3.1* naturally determines a family of Calabi–Yau manifolds, provided that V satisfies a certain combinatorial condition which we shall describe in section five. A hypersurface (holomorphically embedded codimension-one submanifold) $M \subset V$ inherits a Kähler metric by restriction from the embedding in V. M determines a homology class in $H_2(V)$. M is Calabi–Yau precisely when this class is the anticanonical class $-K$. The family of Calabi–Yau manifolds mentioned above is the (finite-dimensional) family of hypersurfaces in this homology class. These are related of course by deformations. They thus represent different complex structures on one underlying differentiable manifold. The significance of this is that the topological **A** models on all of these are isomorphic, this model being independent of complex structure. A submanifold of this type is locally the vanishing locus of a holomorphic function. In a toric variety the homogeneous coordinates z_i allow us to represent M *globally* as the vanishing locus of a homogeneous polynomial $P(z) : Y \to \mathbb{C}$. The Calabi–Yau condition requires that P have degree $\sum_{i=1}^{n} Q_i^a$ under the ath factor of $T = G_{\mathbb{C}}$. The deformations of M (or at least a

subset of the deformations) are described by varying the coefficients of the polynomial P. When V satisfies the combinatorial condition alluded to above, M will be a quasi-smooth hypersurface for generic choices of these coefficients, meaning that M is smooth away from possible singularities of V. If V is smooth, so is M. This condition means, in practice, that the solutions of $dP = P = 0$ (in Y) are contained in F.

We can obtain cohomology classes on M by restriction from classes on V. In general, this will not yield the full cohomology of M but only a subspace, which we will call the "toric" subspace $H^*_V(M)$. In practice, this leads to an extremely simple relation between the intersection form on $H^*_V(M)$ and that on $H^*(V)$. The hypersurface M determines some divisor, and the Calabi–Yau condition on M is equivalent to the requirement that this be the anticanonical divisor

$$-K = \sum_{i=1}^{n} \xi_i \ . \tag{4.1}$$

The ring $H^*_V(M)$ is thus generated by the η_a of the previous section (or rather their intersections with M for which we will use the same notation). The linear relations (3.9) of course still hold. The new intersection form in this presentation is given simply by the standard *restriction formula* for intersections in hypersurfaces:

$$\langle \eta_{a_1} \cdots \eta_{a_p} \rangle_M = \langle \eta_{a_1} \cdots \eta_{a_s} (-K) \rangle_0 \ . \tag{4.2}$$

This leads to a graded ring of length (highest degree in which ring is nontrivial) $d-1 = \dim_{\mathbb{C}} M$. Since we have solved the problem of finding the quantum cohomology ring of V, solving the **A** model with target space M can be thought of as finding the quantum analog of this relation.

Example 1.

A Calabi–Yau hypersurface of $\mathbb{P}^4$ is determined by a generic quintic polynomial $P(z)$. In terms of the homology of V, this determines the class dual to 5η. Thus $H^*_V(M)$ is generated by η subject to the relation $\eta^4 = 0$.

Example 2.

A Calabi–Yau hypersurface is determined by a polynomial of multidegree $(4, 0)$, e.g.

$$P(x) = x_1^8 x_6^4 + x_2^8 x_6^4 + x_3^4 + x_4^4 + x_5^4 + x_1 x_2 x_3 x_4 x_5 x_6 = 0 \ . \tag{4.3}$$

The class determined by this is $4\eta_1$, so that $H^*_V(M)$ is determined by the nonlinear relations (compare (3.14))

$$\begin{aligned} \eta_2^2 &= 0 \\ \eta_1^2(\eta_1 - 2\eta_2) &= 0 \ . \end{aligned} \tag{4.4}$$

4.2. The Model

We now modify the GLSM so that the low-energy theory is the nonlinear sigma model with target space M, following [6]. To this end, we add an additional chiral multiplet Φ_0 with charges

$$Q_0^a = -\sum_{i=1}^{n} Q_i^a \ . \tag{4.5}$$

This modification has the effect of canceling the gauge anomalies in the R-symmetries $Q_{L,R}$ (3.26). This is encouraging; we expect to be obtaining in the infrared a nontrivial fixed point of the renormalization group flow. This should then enjoy $N=2$ superconformal symmetry, which contains left- and right-moving $U(1)$ R-symmetries.

Analyzing this modified model as in section three, we find that the space of supersymmetric ground states is now

$$V^+ = (Y^+ - F^+)/T = (D^+)^{-1}(0)/G \ , \tag{4.6}$$

where $Y^+ = \mathbb{C}^{n+1}$, and F^+ can be read off as in section three from the modified D-terms

$$D_a^+ = -e^2(\sum_{i=0}^{n} Q_i^a |\phi_i|^2 - r_a) \ . \tag{4.7}$$

This determines V^+ as a (noncompact Calabi–Yau) toric variety of dimension $d{+}1$. In fact, (4.5) implies that for r in the Kähler cone $\mathcal{K}_V$ this is the total space of the canonical line bundle over V. This space is the natural setting for Batyrev's mirror construction and will play a central rôle in section five. At this point, however, its salient feature is that it is not M.

To rectify this we make one more modification to the model. The introduction of Φ_0 makes it possible to add a superpotential interaction

$$L_W = -\int d^2z d\theta^+ d\theta^- W(\Phi)|_{\bar\theta^+=\bar\theta^-=0} - \text{h.c.} \tag{4.8}$$

with W a holomorphic, G-invariant function. Note that before introducing the extra field such a function does not exist; it would correspond to a global holomorphic function on V. By contrast, the coefficient of Φ_0^k corresponds to a section of the k-th power of the anticanonical line bundle. A generic choice of W will in fact break the R-symmetry, but if W is homogeneous of some degree in Φ_0 we can recover invariance by accompanying the action of Q_R by a rotation of this superfield. We will choose $W = \Phi_0 P(\Phi)$ where

P does not depend on Φ_0. Then (4.5) shows that this is gauge invariant precisely when $P = 0$ determines a Calabi–Yau hypersurface in V. The nonanomalous R-symmetry is thus directly related to the Calabi–Yau condition. This is natural, as the latter is expected to be the condition for the existence of a nontrivial conformal theory in the low-energy limit.

We can now study the model following the same steps as in section three. We begin with the space of classical ground states. The bosonic potential (3.28) is modified by the addition of

$$U_W = \sum_{i=0}^{n} |\frac{\partial W}{\partial \phi_i}|^2 = |P|^2 + |\phi_0|^2 \sum_{i=1}^{n} |\frac{\partial P}{\partial \phi_i}|^2 \ , \tag{4.9}$$

and by the extension of the D-terms as in (4.7). This affects the space of classical supersymmetric vacua as follows. The first modification to the discussion in section three is that the apparent supersymmetry breaking for r outside the cone $\mathcal{K}_c$ does not arise. The classical moduli space is the entire complexified Kähler space $\mathbb{C}^{n-d}/\mathbb{Z}^{n-d} = \mathbb{R}^{n-d} \times U(1)^{n-d}$. The classical theory is singular along certain cones in r-space found as in section three (without the restriction to $\mathcal{K}_c$), for the same reason, dividing (real) r-space into regions corresponding to different "phases."

Restricting attention once more to the Kähler cone we find that requiring the vanishing of (4.9) in addition to the D-terms will set $\phi_0 = 0$ (restricting to V) and then (4.9) requires that the remaining fields satisfy $P = 0$, in other words that the (point) image of the worldsheet lie in M. This will in fact hold for any $r \in \mathcal{K}_q$. One can study the model more closely and see that the massless modes are precisely the variations of ϕ tangent to this and their superpartners, so we have as the low-energy limit precisely the nonlinear sigma model on the Calabi–Yau hypersurface. The metric on M is classically just the restriction of the metric on V. Notice that with this metric the nonlinear model is not conformally invariant; quantum corrections will presumably shift this to the conformally invariant solution. As is usual, these corrections will not change the Kähler class of the metric, parameterized by t_a. The classical solution determines these in terms of τ as in (3.32); there can be corrections to this as discussed in subsection *3.4*. In other regions of r-space we obtain other types of models. In general these include hypersurfaces in various birational models of V, including models with unresolved orbifold singularities, as well as phases in which the space of vacua is of dimension less than d. In these cases there are massless excitations about these vacua, governed by the superpotential interaction. When the space of vacua is a point the model is what is commonly known as a Landau–Ginzburg

theory, intermediate cases in which there are massless fluctuations about a nontrivial space of vacua were termed "hybrid" models [6]. In many vacua there are discrete subgroups of G unbroken by the expectation values; the low-energy theory is then a quotient by this subgroup. In the Landau–Ginzburg case this leads to the familiar Landau–Ginzburg orbifold models. The physics of hybrid phases is not very well understood. The various theories that arise are classified very naturally by the combinatorics of Δ; equivalently they are obtained from the excluded set F determined by r and the implications of the expectation values required by this when inserted in (4.9). For more details on these, see [6,30,63].

Example 1.

This is treated in detail in [6]. There are two phases separated by a singularity which by (3.41) is located at $r = r_c = \frac{5 \log 5}{2\pi}$ and $\theta = \pi$. For $r \gg r_c$ we find that setting $U = 0$ leads to $\phi_0 = 0$ and $P(\phi = 0)$ as well as $\sum_{i=1}^{5} |\phi_i|^2 = r$. In this region the low-energy limit is the conformal nonlinear sigma model with target space a quintic hypersurface of $\mathbb{P}^4$. For $r \ll r_c$ we see that the vacuum solutions satisfy $\phi_i = 0$, and $|\phi_0|^2 = r$; the vacuum is point-like. Here however the fluctuations of ϕ_i are massless and governed by the superpotential $W = \phi_0 P(\phi)$. This type of model is known as a Landau-Ginzburg theory. Note as well that the choice of expectation value for ϕ_0 does not completely break the gauge symmetry, leaving rather an unbroken $\mathbb{Z}_5$ subgroup. The low-energy theory in this region is thus an orbifold quotient of the LG model by this symmetry.

Example 2.

In the second example, we find the modified D terms

$$\begin{aligned} D_1^+ &= -e^2(|\phi_3|^2 + |\phi_4|^2 + |\phi_5|^2 + |\phi_6|^2 - 4|\phi_0|^2 - r_1) \\ D_2^+ &= -e^2(|\phi_1|^2 + |\phi_2|^2 - 2|\phi_6|^2 - r_2)\ . \end{aligned} \tag{4.10}$$

Let us first find the phase boundaries, by seeking those r values for which an unbroken continuous symmetry is consistent with $D_a = 0$ using (3.28). One finds that g_1 is unbroken if $\phi_3 = \phi_4 = \phi_5 = \phi_6 = \phi_0 = 0$, which from (4.10) can happen at zero energy if $r_1 = 0,\ r_2 \geq 0$. Similarly, g_2 is unbroken if $\phi_1 = \phi_2 = \phi_6 = 0$ which implies $r_2 = 0$ but leads in fact to two cones (rays) because both signs of r_1 are possible. Finally, if ϕ_6 is the only nonvanishing coordinate, then we see that $g_1 g_2^2$ is unbroken. This implies

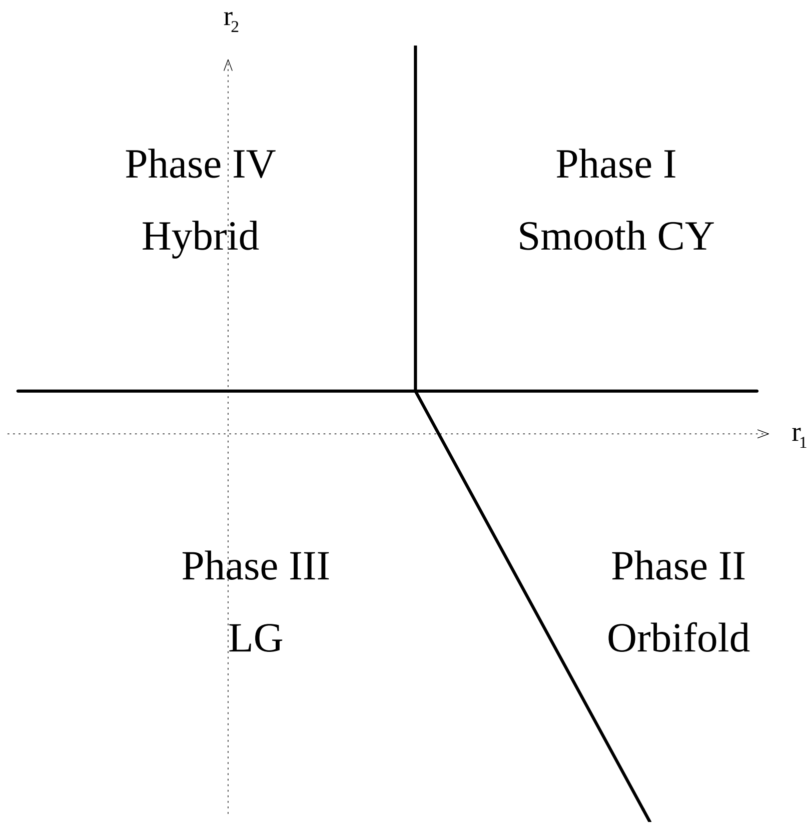

Figure 2. Phase diagram for M.

$r_1 \geq 0,\ 2r_1 + r_2 = 0$. Figure 2 shows the structure in r-space, taking account of the shift (3.41). There are four phases, labeled I–IV.

In phase I we see from (4.10) that the excluded regions are precisely those of (3.4). Requiring the vanishing of U_W then implies $\phi_0 = P = 0$ so the low-energy modes describe the nonlinear sigma model on the Calabi–Yau hypersurface in the smooth toric variety V.

In phase II the excluded regions are $\{\phi_6 = 0\} \cup \{\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0\}$. Once more $U_W = 0$ implies $\phi_0 = 0$. This corresponds to the original (unresolved) projective space; the low-energy limit is the nonlinear sigma model with target space a hypersurface in this space. More precisely we have a deformed version of this because ϕ_6 while nonzero is not fixed to a constant; however for the computations we are interested in this is equivalent. This is the "orbifold" phase.

In phase III the excluded region is $\{\phi_0 = 0\} \cup \{\phi_6 = 0\}$. Then $U_W = 0$ implies the vanishing of all the other coordinates, leading to a unique vacuum configuration in which G is broken to $\mathbb{Z}_2 \times \mathbb{Z}_4$, and to massless fluctuations described by a superpotential interaction. This region thus corresponds to the Landau–Ginzburg orbifold.

Finally, in phase IV the excluded regions are $\{\phi_0 = 0\} \cup \{\phi_1 = \phi_2 = 0\}$. Here $U_W = 0$ implies $\phi_3 = \phi_4 = \phi_5 = \phi_6 = 0$, so that g_1 is broken to a discrete subgroup $\mathbb{Z}_4$. The expectation values of ϕ_1, ϕ_2 parameterize (after setting $D_2 = 0$ and taking the G quotient) a moduli space isomorphic to $\mathbb{P}^1$. The fluctuations of ϕ_3, ϕ_4, ϕ_5 are massless; they interact via a superpotential with coefficients depending upon the point in $\mathbb{P}^1$. The model is a so-called hybrid combining the properties of a sigma model on $\mathbb{P}^1$ with those of a Landau-Ginzburg theory.

The perturbative quantum corrections to this classical picture are found along the lines of the discussion in section three. The novel feature is that (3.39) is satisfied for all a. Thus all of the singularities predicted classically persist in the quantum theory, and occur (far from the origin) at $\theta = 0$ or $\theta = \pi$. This is consistent with the absence of an anomalous R-symmetry which could be used to eliminate one of the θ angles. The only effect of perturbative corrections is the finite shift (3.41) of the asymptotes of the singular locus. Away from the singularities, the low-energy degrees of freedom are the fields Φ_i; the gauge symmetry is Higgsed at generic points throughout parameter space.

In subsection *3.5* we used the twisted superpotential to compute the exact singular locus for the model discussed there. The reasoning presented there can be applied to the model at hand. The results are simpler to describe because the addition of Φ_0 eliminates the distinguished direction in σ-space. Thus, following the arguments presented there, we find that the singular locus is given by the consistency conditions of the equations

$$\prod_{i=0}^{n} \langle \delta_i \rangle^{Q_i^a} = q_a \ . \tag{4.11}$$

Notice that these are all homogeneous of degree zero. As in the previous discussion, this will yield one component of the singular locus (in the present context this is called the *principal discriminant* of the model; note that this is precisely the component absent from the V model of section three because it necessarily involves what was there denoted σ_1). Other components are to be obtained by integrating out subsets of the set of chiral fields charged under subgroups $H \subset G$ such that the charges of the complementary set generate (with positive coefficients) all of $\mathbb{R}^{n-d-k}$ where k is the rank of H. Such a component is given by the consistency conditions for the equations

$$\prod_{i \in I} \langle \delta_i |_H \rangle^{Q_i^a} = q_a \ , \ a = n{-}d{-}k{+}1, \ldots, n{-}d \ , \tag{4.12}$$

where $\{\phi_i\}_{i\in I}$ is the set of charged fields under H, $\{\sigma_a\}_{a=n-d-k+1,\ldots,n-d}$ is the set of massless σ's for H (in an appropriate basis), and where $\delta_i|_H$ is obtained from δ_i by setting $\sigma_a = 0$ for the massive σ_a's, i.e., for $a = 1, \ldots, n-d-k$.

We defer a discussion of the quantum corrections to (3.32) to subsection *4.3*.

Example 1.

In this example the singular divisor – a point, is correctly predicted by (3.41)(modified to include Φ_0), since setting all σ_a but one to zero is the general case. Thus (4.11) here reduces to $q = (-5)^{-5}$.

Example 2.

For the principal component, (4.11) reads (here $s_a = \langle\sigma_a\rangle$ are the expectation values)

$$\begin{aligned} q_1 &= (-4s_1)^{-4} s_1^3 \,(s_1 - 2s_2) \\ q_2 &= s_2^2 \,(s_1 - 2s_2)^{-2} \ . \end{aligned} \tag{4.13}$$

Solutions to these exist when

$$2^{18} q_1^2 q_2 - (1 - 2^8 q_1)^2 = 0 \ . \tag{4.14}$$

The solutions satisfy $s_2/s_1 = (1-2^8 q_1)/2$; this component interpolates between three of the four asymptotic singular limits found in the discussion following (4.10). There are various (rank-1) subgroups of G under which some subset of the fields is uncharged (these were effectively catalogued in constructing the phase diagram for the model). Closer inspection shown that only one of these, however, satisfies the condition that the charges under it span all of $\mathbb{R}$ with positive coefficients. This is the subgroup generated by g_2 as found above, and for it eqn. (4.12) reduces to

$$q_2 = s_2^2 \,(-2s_2)^{-2} = 1/4 \ . \tag{4.15}$$

This is the second component of the singular locus. The corresponding σ-vacua are at $s_1 = 0$. They occur for any value of q_1, of course. Figure 3 shows the exact singular locus in the r-plane. The curve shows the singularity for real q; singularities at other values of θ exist for r in the enclosed region in the center of the figure.[17] Asymptotically at large

[17] This figure was first calculated in [60] for the B-model of the mirror partner; here we get it directly from A-model considerations.

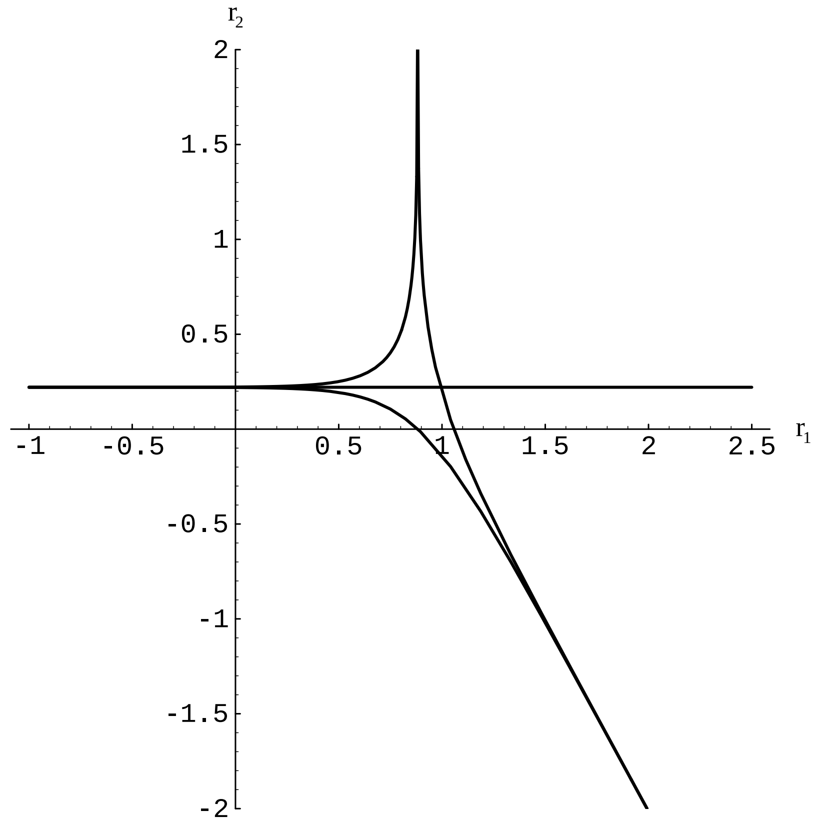

Figure 3. Fully corrected phase diagram for M.

r the singular locus approaches the cones predicted in subsection *3.2* above. For finite r there are instanton corrections as predicted in section two and computed above; these introduce a θ dependence in the singular locus (which is a holomorphic divisor in q space).

For the remainder of this section we once more restrict attention to the "geometric" phases (values of r in the interior of $\mathcal{K}_q$). (This includes the phases we have labeled I and II in example 2.)

4.3. The Topological Model

As in the previous section, our main interest will be in the topological **A** model. Our earlier conclusions regarding the spectrum of Q-closed local operators are still valid. The quantum cohomology algebra is generated by σ_a. This is in line with the description of the cohomology ring $H^*_V(M)$ in subsection *4.1*. The correlation functions we compute will (for r in $\mathcal{K}_V$) be interpreted as the correlators in the nonlinear sigma model with target space M. We denote the correlation functions in the new model by

$$X_{a_1\ldots a_s} = \langle\!\langle \sigma_{a_1}(z_1)\cdots\sigma_{a_s}(z_s)\rangle\!\rangle \ . \tag{4.16}$$

It is easy to see that the interaction (4.8) is in fact Q-exact. Thus our correlation functions will be independent of the coefficients in the polynomial P.

In subsection *3.6* we found that the twisted superpotential $\widetilde{W}(\Sigma)$ computed at one-loop order about the free gauge theory at large σ led to operator relations (3.44) which held throughout parameter space by analytic continuation. In the case at hand, however, this is not *a priori* clear, because there is no domain from which to continue analogous to the interior of the complement of $\mathcal{K}_c$. When we compute the correlators, however, we will find that in fact the relations (4.11) hold exactly in all correlators, and in the case of smooth V suffice to determine them. Of course, away from the singular locus (4.11) are relations between the operators δ_i and not their expectation values. This point will be discussed more fully in section five.

The ghost number conservation law (3.48) will be altered by the superpotential term in the action. The modified R-symmetry under which the superpotential interaction is invariant will assign unit R-charge to ϕ_0 (of course this assignment is determined up to the possibility of an accompanying gauge transformation, which will have no effect on the discussion). Computing the contribution of this to the gravitational anomaly (3.48) we find indeed[18]

$$\Delta(Q_R) = -\Delta(Q_L) = \frac{d-1}{2}\chi(\Sigma) \ , \tag{4.17}$$

where the instanton contribution vanishes by (3.39). This agrees with our observations in the zero instanton sector where we found a restriction to $M \subset V$. The correlation functions neglecting instanton contributions will give the intersection form on M as determined from (4.2). To compute instanton corrections to this we now proceed to study how equations (3.54) are modified.

To begin with, of course, there is an additional field ϕ_0. Further, the instanton equations are modified. The additional condition imposed is

$$\frac{\partial W}{\partial \phi_i} = 0 \ . \tag{4.18}$$

This is invariant under local $G_{\mathbb{C}}$ transformations so the discussion following (3.54) is still valid. In particular, before imposing (4.18) the equations (3.54) (including ϕ_0) still determine a toric variety. In fact, this will be precisely $\mathcal{M}_{\vec{n}}$, as we now show. The argument

[18] A detailed discussion of the R-symmetry in the model and its relation to the superconformal symmetry of the low-energy limit has been given in [64].

surrounding (3.63) shows that for $r \in \mathcal{K}_V$, the moduli space is empty unless $\vec{n} \in \mathcal{K}_V^\vee$. For these values of $\vec{n}$ we have $d_0 = \sum_a Q_0^a n_a = -\sum_{i=1}^n d_i \leq 0$. Thus ϕ_0 is identically zero or at most a constant. But if ϕ_0 is a nonzero constant then the image in V of the map (forgetting ϕ_0) is constrained by (4.18) to lie in the critical point set of P (as a function on Y). We will use our freedom to modify the coefficients of P to choose it such that $P = 0$ is a *quasi-smooth* hypersurface in V. This condition – which holds for generic values of the coefficients – means that the critical points of P are contained in the set F. This however implies that if ϕ_0 is nonzero then the image is contained in F and so such maps are not (complex) gauge transforms of true solutions. On the other hand, setting $\phi_0 = 0$ we recover the original equations (3.54) for ϕ_i. In all, the moduli space from which we obtain the instanton contributions to correlation functions is the subset satisfying (4.18) in $\mathcal{M}_{\vec{n}}$.

Setting $\phi_0 = 0$ this becomes the equation

$$P(\phi) = 0 \ . \tag{4.19}$$

The homogeneity properties of P guarantee that it is a section of a line bundle over $\mathbb{P}^1$ of degree $-d_0 = \sum_{i=1}^n d_i$, hence described by a homogeneous polynomial of this degree in (s,t). Solving (4.19) identically is the same as requiring that all $1-d_0$ coefficients of this polynomial vanish. These coefficients, in turn, are polynomials in the homogeneous coordinates ϕ_{ij} on $\mathcal{M}_{\vec{n}}$. For generic maps these will be $1-d_0$ independent equations in ϕ_{ij}, so the formal dimension (an estimate of the true dimension) of the moduli space of solutions to the full set of equations (compare (3.61)) is just $d-1$, the dimension of M, and independent of n as expected. As before, this space is a compactification of the space of maps $\Sigma \to M$ obtained by adding maps that intersect $F \cap \widehat{M}$, where $\widehat{M} \subset Y$ is the vanishing locus of P interpreted as a function on $Y = \mathbb{C}^n$.

The nonvanishing correlation functions are now of the form (4.16) with $s = d-1$. Interpreting the $d-1$ inserted operators as restricting the path integral to the intersection of $d-1$ divisors in $\mathcal{M}_{\vec{n}}$, we find a finite number of solutions at each instanton number. In the model of section three the contribution of the $\vec{n}$th instanton sector was computed simply by counting these solutions. The new feature here is that the contribution counts solutions with a nontrivial sign. In the absence of a superpotential interaction (and of an expectation value for the σ fields) the fermion determinant is manifestly real and positive because the (λ^+, ψ^-) determinant is the complex conjugate of the (λ^-, ψ^+) determinant.

We can choose the sign of the operator insertions to respect this complex structure. The superpotential, on the other hand, manifestly reverses the complex structure, because it leads for example to a $\psi^i_-\psi^j_+$ term. In computing the contribution at instanton number $\vec{n}$ to some nonzero correlator there are $1-d_0$ zero modes of ψ_0 which cannot be soaked up in the operator insertions (which only involve χ and $\bar{\chi}$, or in other words have positive (negative) charge under the right-moving (respectively left-moving) R-symmetry. These zero modes are lifted by the superpotential term; the resulting measure on the remaining zero modes is the product of a positive quantity (essentially $|\partial^2 W|^2$ evaluated on the zero modes) and the sign originating from the reordering of the fermionic integral required to render the coefficient of this term positive. Thus the contribution of $\mathcal{M}_{\vec{n}}$ is weighted by a sign proportional to $(-1)^{1-d_0}$. (The two extra χ zero modes are more subtle to treat, but will not affect this discussion.) The overall sign of the series in q is of course a matter of convention, and we fix it to obtain agreement with the classical result at $q=0$. This leads to the final result that the contribution of $\mathcal{M}_{\vec{n}}$ is weighted by $(-1)^{d_0}$.

We have mentioned above that the noncompactness of instanton moduli spaces and the need to carefully account for "contributions at infinity" are a serious obstacle to computing Gromov–Witten invariants. Our moduli spaces are compact, but these problems reappear in the current context as a degeneration of the instanton equations at the compactification subsets of $\mathcal{M}_{\vec{n}}$. (This was pointed out in [6].) The naïve dimension counting fails precisely for "pointlike" instantons. Heuristically, the common factor in the polynomials which characterizes such maps means that P is effectively of lower degree, and the space of maps satisfying $P=0$ is of larger dimension. This means we cannot think of adding the pointlike instantons to the space of holomorphic maps $\Sigma \to M$ as a compactification by adding lower-dimensional spaces; at a given degree the generic map is pointlike. For this reason we cannot expect the contribution to the correlation function at any given degree as we compute it to be the same as the contribution at this degree in the nonlinear sigma model. However, our arguments show that the low-energy limit of the theory is a nonlinear sigma model with target space M, and that further the correlation functions computed in the microscopic theory should be equal to those in the low-energy theory. The resolution to this apparent contradiction is that including the pointlike instantons can lead to a renormalization of the couplings in the low-energy model. In the case at hand these are the τ_a. In other words, we can expect the correlation functions we compute to reproduce the correlators in the nonlinear sigma model, but (3.32) will receive nontrivial corrections in this case. The finite shift from perturbative effects allowed for there is computable

in this case. The contributions of smooth instantons in the nonlinear model are counted with positive signs (for essentially the reason given above in the absence of superpotential interactions), whereas we have here found a nontrivial sign. This is interpreted as a finite shift in θ correcting (3.32)

$$t_a = \tau_a + \frac{1}{2} Q_0^a \ . \tag{4.20}$$

In addition, there will be corrections due to the contributions of pointlike instantons. As we have seen, the moduli space of pointlike instantons of instanton number $\vec{n}$ is isomorphic to the space of instantons of a lower degree. Adding such lower degree corrections at degree $\vec{n}$ is the typical effect of corrections to (4.20) with exponential drop-off at large r. This will be borne out in the examples in the next subsection; we note that this situation differs from that encountered in section three. Thus, in terms of the Gromov–Witten invariants, we will obtain the correct generating function but not the correct expansion variable.

In fact, we will not use the instanton moduli space in the form described above. We find it more convenient to choose a different way of incorporating the effects of the pointlike instantons. This will also lead to nothing more than a renormalization of the couplings in the low-energy Lagrangian. We simply replace (4.19) by the requirement that $P = 0$ hold at some fixed collection of points $Q_0, \dots Q_{-d_0}$ in Σ (with $1{-}d_0$ points in all). This is the same as (4.19) for non-pointlike maps. The advantages of this particular choice are immediately apparent. We have effectively reduced the computation once more to a set of intersection calculations in the toric varieties $\mathcal{M}_{\vec{n}}$, a manifestly tractable problem. Writing an instanton expansion for the correlation functions as before we are claiming that the modified (3.70) reads

$$\begin{aligned} X^{\vec{n}}_{a_1 \dots a_s} &= (-1)^{d_0} \langle (\eta_{a_1})_{\vec{n}} \cdot (\eta_{a_2})_{\vec{n}} \cdot \dots \cdot (\eta_{a_s})_{\vec{n}} \chi_{\vec{n}} (-K)^{1-d_0}_{\vec{n}} \rangle_{\vec{n}} \\ &= -\langle (\eta_{a_1})_{\vec{n}} \cdot (\eta_{a_2})_{\vec{n}} \cdot \dots \cdot (\eta_{a_s})_{\vec{n}} \chi_{\vec{n}} (K)^{1-d_0}_{\vec{n}} \rangle_{\vec{n}} \ , \end{aligned} \tag{4.21}$$

where $(K)_{\vec{n}}$ is obtained from (4.1) using the usual lift to $\mathcal{M}_{\vec{n}}$. This equation contains a complete solution to the **A** model with target space M.

The fact that the ghost number anomaly is independent of instanton number means that (4.21) gives the Hilbert space the structure of a graded algebra of length $d{-}1$. Put otherwise, it is clear that (4.21) vanishes unless $s = d{-}1$; when this holds there will be nonzero contributions from all instanton sectors in $\mathcal{K}_V^\vee$.

The correlation functions computed by summing (4.21) will be algebraic functions of the q_a. In particular, their only singularities will be poles. We will explicitly see

this in subsection *4.5*. This property singles out the particular set of coordinates and frame in which we are working (as well as the overall normalization factor – the gauge choice – arising from the normalization of the vacuum state). We call these the *algebraic coordinates* and *algebraic gauge*. As anticipated above, these differ from the canonical coordinates on moduli space. However, the algebraic coordinates have the distinguishing property that they are "good" coordinates globally. In terms of these coordinates, one can unambiguously continue the correlation functions around their poles and into regions of the moduli space described by "nongeometric" phases of the theory. The analytic properties of correlation functions in topological field theories away from the singularities mean that these analytic continuations in fact yield the correct correlation functions in these regions of parameter space. In section five we present an approach to the model which allows us to independently compute the expansions about nongeometric limiting points.

There are also algebraic coordinates which appear naturally in discussions of **B** models. The correlation functions of these are naturally expressed in terms of intrinsically algebraic objects, e.g., local rings of polynomials. In [39] a mapping from these coordinates (in terms of the **B** model on a manifold W) to the canonical coordinates (on the moduli space of the **A** model on its mirror M), valid in the large-r limit, was proposed and named the *monomial-divisor mirror map*. What we have given above is in fact an interpretation of the algebraic coordinates in the interior of **A** model moduli space. They represent the set of coordinates and the frame that arise naturally in the GLSM. To give a more complete understanding of this directly in terms of the low-energy nonlinear model we should compute the instanton corrections to (4.20). We believe this should be possible using the simple recursive structure of $\mathcal{M}_{\vec{n}}$, but will not pursue the issue here. In what follows we will use the natural GLSM coordinates; to obtain from these the *algebraic coordinates* for the low-energy nonlinear model (in the terminology of [39]), the shift (4.20) should be applied.

4.4. Solving the Examples

In this subsection we illustrate the construction of the previous subsection by explicitly solving our two Calabi–Yau examples. Since we have $d = 4$, the nonzero couplings we compute will be the trilinear Yukawa couplings. In the first example we have precisely one of these; it is

$$X_3 = \langle\langle \sigma^3 \rangle\rangle = \sum_{n\geq 0} X_3^n q^n \ . \tag{4.22}$$

Here we have $d_0(n) = -5n$ and $K = -5\eta$, and the coefficients in the expansion are

$$X_3^n = -\langle \eta^3 K^{5n+1} \rangle_n = -\langle \eta^3 (-5\eta)^{5n+1} \rangle_n = -(-5)^{5n+1} \ . \tag{4.23}$$

Inserting this we can sum the series to find

$$X_3 = \frac{5}{1 + 5^5 q} \ . \tag{4.24}$$

This exhibits the expected singularity at $\tau = \frac{1}{2} + i\frac{5\log 5}{2\pi}$. Notice, however, that the singularity is just a simple pole in q and there is no difficulty in continuing around this pole to define the correlation function for all r, θ. The Yukawa coupling in this model has been computed by Candelas et al. [38] using mirror symmetry; our result is in complete agreement with theirs if one sets $q = (-5\psi)^{-5}$ and $X_3 = \kappa_{\tau\tau\tau}$ and performs the change of variables treating κ as a tensor. (Note that in fact κ is a tensor-valued section of a line bundle $\mathcal{L}^2$. We have obtained a trivialization of $\mathcal{L}$ identical to that which came up naturally in the work of [38]. We will discuss this issue in section five.)

In the second example there are four different Yukawa couplings, which we denote by

$$X_j = \langle\!\langle \sigma_1^{3-j} \sigma_2^j \rangle\!\rangle \ . \tag{4.25}$$

Here we have $d_0(n) = -4n_1$ and $K = -4\eta_1$; the expansion coefficients are thus

$$X_j^{\vec{n}} = -\langle \eta_1^{3-j} \eta_2^j K^{4n_1+1} \chi_{\vec{n}} \rangle_{\vec{n}} = 2^{8n_1+2} \langle \eta_1^{4n_1+4-j} \eta_2^j \rangle_{\vec{n}} \ . \tag{4.26}$$

Comparing (3.76) we thus find

$$X_j = X_j^{(0)} + \sum_{n_1, n_2 \geq 0; (n_1, n_2) \neq (0,0)} 2^{8n_1+2n_2+3-j} \begin{pmatrix} n_1 + 1 - j \\ 2n_2 + 1 - j \end{pmatrix} q_1^{n_1} q_2^{n_2} \ . \tag{4.27}$$

We have separated the $\vec{n} = 0$ term out; this is the classical intersection calculation on M, given by (4.2). Performing the sums we find

$$\begin{aligned} X_0 &= \frac{8}{\Delta} \\ X_1 &= \frac{4(1 - 2^8 q_1)}{\Delta} \\ X_2 &= \frac{8q_2(2^9 q_1 - 1)}{(1 - 4q_2)\Delta} \\ X_3 &= \frac{4q_2(3072 q_1 q_2 + 2^8 q_1 - 4q_2 - 1)}{(1 - 4q_2)^2 \Delta} \ , \end{aligned} \tag{4.28}$$

where $\Delta = (1-2^8 q_1)^2 - 2^{18} q_1^2 q_2$ (compare (4.14) and (4.15)).

Comparing with the correlation functions computed in the mirror model by Candelas et al. [45], we find that under the substitution $(-\phi)^{-2} = 4q_2$, $(-\psi)^{-8} = 2^{24} q_1^2 q_2$ the correlation functions are identical if thought of as the components of a tensor of rank three on the parameter space. (This substitution is precisely the one specified by the monomial-divisor mirror map [39], as noted in [45].)

These results verify our conclusion that we have correctly summed the instanton series to obtain exact correlation functions. They also show that these are obtained in the "algebraic" coordinates and gauge.

4.5. Quantum Restriction Formula

In the previous subsections we have shown how toric geometry allows us to sum the instanton series and compute exactly the correlation functions for the topological nonlinear sigma model with target space M. In this subsection we will find a general formula relating the quantum cohomology of M to that of V. In the case of smooth V, where the discussion of subsection *3.9* gives an algebraic computation of the latter, the result below leads to an algebraic computation for M. Essentially we will find the quantum version of (4.2).

We have almost found this in the formula (4.21). We would like to interpret this formula as the contribution (3.70) to some correlator in the V theory. The difficulty is only that the operator inserted in (4.21) seems to depend upon $\vec{n}$ (beyond the simple lift in (3.70)). The solution to this is quite simple, however. When (4.21) is nonzero and $s = d-1$ is the dimension of M – in which case the correlator we are computing receives nonzero contributions from all instanton numbers – we can in fact replace (4.21) by

$$X^{\vec{n}}_{a_1 \ldots a_s} = -\Big\langle (\eta_{a_1})_{\vec{n}} \cdot (\eta_{a_2})_{\vec{n}} \cdot \cdots \cdot (\eta_{a_s})_{\vec{n}} \chi_{\vec{n}} \sum_{m>0} K^m_{\vec{n}} \Big\rangle_{\vec{n}} \ . \tag{4.29}$$

Because $d_0 \leq 0$ one term in the sum over m is (4.21); all other terms vanish because the class we evaluate is not of top degree in $\mathcal{M}_{\vec{n}}$. Now we can regroup the sums in $X_{a_1 \ldots a_s}$ to write the correlation function in the M theory as a particular computation in the quantum cohomology ring of V

$$X_{a_1 \ldots a_s} = -\sum_{m>0} \langle \sigma_{a_1} \cdots \sigma_{a_s} K^m \rangle \ . \tag{4.30}$$

Notice that if $s < d-1$ then (4.29) and (4.30) still coincide with (4.21), since all are zero; however, if $s > d-1$ then (4.29) and (4.30) can be nonzero even though (4.21) vanishes.

More mathematically, thinking of the correlation function as defining an expectation function on the quantum cohomology algebra, we can write a general result, the *quantum restriction formula*, for the restriction to an anticanonical hypersurface:

$$\langle\langle \mathcal{O} \rangle\rangle = \langle \mathcal{O} \frac{-K}{1-K} \rangle \tag{4.31}$$

where $\mathcal{O}$ is any polynomial in η_a of degree at most $d-1 = \dim_{\mathbb{C}} M$. This provides a way to compute the quantum cohomology algebra of M in terms of the quantum cohomology of V, the latter being a problem which was solved in the previous section. This quantum restriction formula is to be interpreted as follows: the sum

$$\sum_{m=1}^{\infty} K^m \tag{4.32}$$

should be expected to converge in the quantum cohomology algebra of V (with respect to any Hermitian norm on that algebra, thought of as a complex vector space). If it does converge as expected, then $1-K$ must be invertible in the algebra, and the sum (4.32) must converge to $-K/(1-K)$. The correlation functions for M can thus be computed using (4.31). (Although we cannot directly interpret (4.31) as defining an expectation function on a quotient algebra, we will see in the next section that a variant of this calculational procedure does admit such an interpretation.)

We note that when $-K = pe$ for $e \in \mathbb{Z}[\eta_1, \ldots, \eta_{n-d}]$ then the summand in (4.29) is nonzero (at any $\vec{n}$) only for $m = 1 \bmod p$, so (4.31) can be replaced by

$$\langle\langle \mathcal{O} \rangle\rangle = \langle \mathcal{O} \frac{-K}{1-K^p} \rangle \; ; \tag{4.33}$$

this form is manifestly consistent with the $\mathbb{Z}_p$ grading and is useful for explicit computations. Eqn. (4.31) (or its variant eqn. (4.33)) is precisely the desired quantum version of (4.2).

Example 1.

We work in the quantum cohomology ring of $V = \mathbb{P}^4$, which is generated by σ with relation $\sigma^5 = q$; the expectation function is normalized by $\langle \sigma^4 \rangle = 1$. Since $-K = 5\sigma$, we may use (4.33) with $p = 5$, and we thus must calculate

$$\frac{-K}{1-K^5} = \frac{5\sigma}{1-(-5\sigma)^5} = \frac{5\sigma}{1+5^5 q} \; . \tag{4.34}$$

From this, it is easy to check that the quantum restriction formula (4.33) agrees with the earlier calculation (4.24).

Example 2.

In this example, the quantum cohomology ring of V is generated by two classes σ_1 and σ_2, and the ideal of relations is generated by the two polynomials

$$F_1 \equiv \sigma_1^3(\sigma_1 - 2\sigma_2) - q_1 \ , \tag{4.35}$$

$$F_2 \equiv \sigma_2^2 - q_2(\sigma_1 - 2\sigma_2)^2 \ . \tag{4.36}$$

The normalization of the expectation function can be taken as $\langle\sigma_1^4\rangle = 2$ or equivalently $\langle\sigma_1^3\sigma_2\rangle = 1$ (as is seen from (3.72) with $\vec{n} = 0$), and the anticanonical class is $-K = 4\sigma_1$. We can set $p = 4$ in (4.33); in order to find

$$\frac{-K}{1-K^4} = \frac{4\sigma_1}{1-4^4\sigma_1^4} \tag{4.37}$$

we must make some calculations with the ideal generated by F_1 and F_2. The specific element of that ideal which we need is

$$\left[(1+4q_2)\sigma_1^4 + (2-8q_2)\sigma_1^3\sigma_2 + (4q_1q_2 - q_1)\right] F_1 + 4\sigma_1^6 F_2 = \sigma_1^8 - 2q_1\sigma_1^4 + q_1^2(1-4q_2) \ . \tag{4.38}$$

From the fact that the right side of (4.38) is zero in the quantum cohomology ring, it is easy to find the inverse of $1 - 4^4\sigma_1^4$ in that ring and to conclude that

$$\frac{4\sigma_1}{1-4^4\sigma_1^4} = \frac{4\sigma_1(2^8\sigma_1^4 + (1-2^9q_1))}{\Delta} \ , \tag{4.39}$$

where $\Delta = (1-2^8q_1)^2 - 2^{18}q_1^2q_2$ as in eqn. (4.28). A short calculation then verifies that the quantum restriction formula (4.33) reproduces eqn. (4.28).

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5. The V^+ Model, Mirror Symmetry and the Monomial-Divisor Mirror Map

In the previous section we have, essentially, solved the **A** topological nonlinear sigma model with target space M a hypersurface in a toric variety V. Our computation was based in the Kähler cone of V, but we found that the correlation functions are meromorphic objects, hence naturally determine analytic continuations to the entire parameter space of the model. This is certainly natural from the point of view of the mirror theory, since

the space of complex structures on the mirror manifold W is of the general form $A - B$ where A and B are subvarieties in projective space (the rôle of B is played by the singular locus, A is essentially complexified r-space). It thus becomes unnatural to treat the theory in a way that distinguishes the Kähler cone. In this section we reformulate the model so that instanton expansions may be computed about a limiting point deep in the interior of *any* of the cones in r-space, leading to analytic correlation functions throughout the nonsingular part of the parameter space.

It is intriguing that hypersurfaces in toric varieties are precisely the class of Calabi–Yau manifolds for which a detailed conjecture on the construction of a mirror manifold W, itself realized as a hypersurface in a toric variety Λ, was proposed by Batyrev [26]. Since in some sense at least the **B** model on W should be easier to solve than the **A** model on M (the correlation functions are given in terms of classical geometry) we should be in a position to prove these conjectures, at the level of topological field theory. Furthermore, since we have explicit solutions we should be able to shed some light on the mirror map between the moduli spaces of the two models. In this section we will partially fulfill these expectations. In particular, in the case that V is smooth we have an algebraic computation of the correlators. It is in this case that we will manage to make contact with **B** model calculations. Our ability to prove the conjecture in this class of models is limited only by the absence of a general algebraic solution of the **B** model.

5.1. Toric Geometry, Once Again

Given a toric variety V we have seen that there is a canonical way of producing a Calabi–Yau hypersurface in V (more properly, a family of hypersurfaces related by deformations). There is another way of producing a Calabi–Yau space, from any Kähler manifold. We consider not a hypersurface but the total space of a line bundle over V. To have trivial canonical class this line bundle should be the canonical line bundle. For a toric variety V this construction was introduced in section four as V^+. This manifold is simple in that it is a *toric* Calabi–Yau manifold. Of course, it is noncompact. A holomorphic quotient construction of V^+ is given by (4.6).

In terms of the fan Δ we obtain a toric construction for V^+ by considering a fan Δ^+ in $\mathbb{R}^{d+1}$ given by $(n+1)$ vectors lying in the affine hyperplane $y_{n+1} = 1$. These comprise the vertices of Δ, promoted to the hyperplane by the addition of a last component whose value is 1, as well as the origin in the construction of Δ, similarly promoted. As is clear

from the affine hyperplane condition, this is not a complete fan, as expected since V^+ is noncompact.

The dimension of V^+ is $d+1$. The problem of computing the ring structure on the cohomology of V^+ is ill-posed since the manifold is noncompact, but there is certainly one compact divisor – the zero section of the bundle $\{x_0 = 0\} = V$. If we only consider intersection calculations involving this divisor (and other classes) then the problem will be well-posed. The class of this divisor is represented by ξ_0, and in practice,

$$\mathrm{Tr}_{\mathrm{cl}}(\mathcal{O}) = \langle \xi_0 \mathcal{O} \rangle_{V^+} \tag{5.1}$$

defines an expectation function on a quotient algebra $\mathbb{C}[\eta_1, \ldots, \eta_{n-d}]/\mathcal{J}_{\mathrm{cl}}$ (by Nakayama's theorem), yielding a graded ring of length d. Of course, this will be precisely $H^*(V)$. As in the previous section, we will look for a useful quantum generalization of this. What we will find is in fact a modification of the quantum cohomology of V, one which leads very naturally to a solution for the quantum cohomology of the hypersurface M.

The combinatorial condition [26] which ensures that V contains quasi-smooth Calabi–Yau hypersurfaces M is most easily stated in terms of V^+ and Δ^+. Consider the cone $\Sigma = |\Delta^+|$ which is the support of Δ^+ in $\mathbf{N}^+_{\mathbb{R}} \sim \mathbb{R}^{d+1}$. This is a *Gorenstein* cone (cf. [28]), which means if we take its intersection with the hyperplane $n_{d+1} = 1$ we obtain points of the lattice $\mathbf{N}$ (the generators of one-dimensional cones in Δ) as vertices of the resulting polytope $\mathcal{P} \equiv \Sigma \cap \{n_{d+1} = 1\}$. The condition we need is that Σ be a *reflexive* Gorenstein cone of index one.[19] This means that if we construct the dual cone $\Sigma^\vee = \{m \in \mathbf{M}^+_{\mathbb{R}} \mid m \cdot n \geq 0 \, \forall n \in \Sigma\}$, it too must be Gorenstein, that is, its intersection with the hyperplane $m_{d+1} = 1$ must be a polytope $\mathcal{P}^0$ whose vertices lie in the lattice $\mathbf{M}$. $\mathcal{P}^0$ is called the *polar polytope* of $\mathcal{P}$.

We now review the construction, due to Batyrev [26], of a dual (to V) toric variety Λ. Batyrev's conjecture is that the family of Calabi–Yau hypersurfaces W determined by Λ is related by mirror symmetry to M. More accurately, the facet of the conjecture upon which we focus here states that some subspace of the space of deformations of W is related to the space of Kähler structures (r-space) for M by a mirror map, which will equate correlation functions in the **A** model for M as computed above to correlators in the **B** model on W.

We can use the polar polytope $\mathcal{P}^0$ to determine Λ. To do this we need to find the fan ∇ which will play the rôle that Δ played for V, so we need to see the relation between

[19] This version of the definition from [28] is equivalent to Batyrev's original definition in [26].

$\mathcal{P}$ and Δ. The one-dimensional cones in Δ include the rays containing the vertices of $\mathcal{P}$, of course. The construction guarantees that both $\mathcal{P}$ and $\mathcal{P}^0$ have a unique interior lattice point – the origin in $\mathbf{N}$ and $\mathbf{M}$ respectively. But there can be points in the faces of the polytope, and we can choose different fans by including different subsets of the rays generated by these in the set of one-dimensional cones. As described in detail in [30], different choices correspond to different birational models; in our language different choices for Δ correspond to different values for r in the cone $\mathcal{K}_q$ of "geometric" phases. There is thus some ambiguity in determining Λ, but for our current purposes this will be irrelevant. The reason is that we will be interested in the $\mathbf{B}$ model on a hypersurface $W \subset \Lambda$. For this model, the parameters r are irrelevant (they couple to Q-exact operators) just as the coefficients of the polynomial P were irrelevant to the $\mathbf{A}$ model on $M \subset V$. We will find it convenient to choose a model for Λ in which we include none of these extra points. This (generically singular) toric variety has the property that its anticanonical divisor class is ample. The salient point here is that Λ is determined uniquely by this construction to within irrelevant choices. Then, as described in section four, Λ itself determines a family of Calabi–Yau hypersurfaces $W \subset \Lambda$ related by deformations. The deformation space of these will be related by mirror symmetry to the r-space of the $\mathbf{A}$ model on M. The family of all models corresponding to any point in r-space is determined by the polyhedron $\mathcal{P}$ (which is the convex hull of the integral generators of the one-dimensional cones in the fan Δ); this is also precisely the information needed to construct Σ and hence Λ. This shows that the construction is involutive in the sense that had we started with the fan ∇ for some model of Λ we would have been led to construct $\mathcal{P}^0$ and the dual cone would lead to $\mathcal{P}$. When studying the $\mathbf{A}$ model, we will prefer to find a smooth model for V (we assume such a model exists) and so will include the rays generated by *all* of the points of $\mathcal{P}$ in Δ.[20]

The rather abstract combinatorics discussed above can be interpreted in a somewhat more concrete way as follows. As mentioned in previous sections, points in $\mathbf{M}$ correspond to monomials in the homogeneous coordinates which can be interpreted as meromorphic functions on V. Points in $\mathbf{M}^+$ can (by including the appropriate powers of x_0 with the monomial) be seen to correspond to sections of powers of the anticanonical line bundle. The

[20] The exception to this is the case of points in the interior of codimension-one faces of $\mathcal{P}$. These correspond to point singularities in V which therefore will be disjoint from generic hypersurfaces; we do not need to include these.

cone $\Sigma^\vee$ contains the *holomorphic* sections; thus the set of all points in ∇ is precisely the set of holomorphic sections of $-K_V$. This was identified in section four as being precisely the set of possible gauge-invariant monomials linear in Φ_0 which could be used to construct the superpotential W. The involutive property of the construction discussed above means we can give the same interpretation to the points in $\mathcal{P}$ as monomials appearing in the superpotential for the GLSM describing $W \subset \Lambda$. This association of divisor classes (ξ_i) to the corresponding monomials – call them μ_i – is the basis for the *monomial-divisor mirror map* of [39], and will appear in our discussion of mirror symmetry.

5.2. The Topological Model for V^+

When we studied the gauged linear sigma model leading to the M model we found that the correlators of this theory are independent of the details of the superpotential W. It is tempting, therefore, to simplify the model by simply setting $W = 0$.[21] By the discussion of section two this is naïvely expected (for some range of r) to lead to a topological theory equivalent to the **A** model with target space V^+. Of course, there are difficulties with defining this model, because of the noncompact space of bosonic zero modes; the point $W = 0$ is clearly a singular point in the parameter space. Further, if we were to obtain the V^+ model it would be a topological conformal field theory (since V^+ is Calabi–Yau) with central charge (ghost number anomaly) $d+1$ – the dimension of V^+ – rather than $d-1$. We will see that the two theories, while not identical, are intimately related.

We thus proceed to apply the formal methods of section three to the model of the previous section with $W = 0$, which we will call the V^+ theory. The absence of a superpotential will suggest a more symmetric treatment of ϕ_0 and the ϕ_i. The Q-cohomology classes are still represented by the σ_a which we continue to identify with η_a. The path integral will be well-defined at the level of zero modes for correlation functions containing an insertion of

$$\delta_0 \sim -\sum_{a=1}^{n-d} Q_0^a \sigma_a = -\sum_{i=1}^{n} \xi_i \ . \tag{5.2}$$

The zero section $V \subset V^+$ is a representative of the divisor class ξ_0. Thus in the zero mode sector this insertion of δ_0 reduces the integral to V. In the presence of this insertion, the discussion of the model along the lines of previous sections is straightforward. Once more

[21] We thank E. Witten for suggesting that the computations from section four might have an interpretation in such a theory.

one finds singularities along the cones in r-space on which a continuous symmetry is unbroken; perturbative quantum effects are limited to the finite shift (3.41). The computation of the singular locus in section four made no use of the superpotential so continues to hold here as well. The new features arise when we consider instanton effects.

In this model the only modification to (3.54) is the addition of the extra field ϕ_0. This in effect means that the moduli spaces are naturally obtained as noncompact toric varieties, at least when $d_0 \geq 0$. If we restrict attention momentarily to the "geometric" phases, however, the presence of δ_0 in correlation functions will suffice to reduce the computation to an intersection computation in the compact moduli spaces $\mathcal{M}_{\vec{n}}$ of the previous sections. The arguments immediately following (4.18) show that $d_0 \leq 0$. Thus ϕ_0 is at most a constant; then the insertion of $(\delta_0)_{\vec{n}}$, interpreted as restricting attention to maps for which ϕ_0 vanishes at a point, is in effect a restriction to $\phi_0 = 0$. Thus in the "geometric" phases, the instanton moduli spaces are the $\mathcal{M}_{\vec{n}}$ of section three. The correlation functions we compute will be denoted

$$Z_{a_1 \ldots a_s} = \langle\langle\langle \sigma_{a_1}(z_1) \cdots \sigma_{a_s}(z_s) \delta_0(z_0) \rangle\rangle\rangle \ . \tag{5.3}$$

By the reasoning of previous sections these have an expansion

$$Z_{a_1 \ldots a_s} = \sum_{\vec{n} \in \mathcal{K}^\vee} Z^{\vec{n}}_{a_1 \ldots a_s} \prod_{a=1}^{n-d} q_a^{n_a} \ . \tag{5.4}$$

The restriction to $\mathcal{K}^\vee$, the dual of the Kähler cone of V, follows here using the δ_0 insertion.

The expansion coefficients are once more computed as intersection calculations in $\mathcal{M}_{\vec{n}}$. Since $\phi_0 = 0$ is here imposed by degree considerations and not by (4.18) we will have ghost zero modes and an associated Euler class. In addition, due to the absence of a superpotential the sign that modified the contribution computed in section four is absent here. The arguments of section three lead, up to normalization, to

$$\chi^+_{\vec{n}} = \prod_{d_i < 0} (\xi_i)_{\vec{n}}^{-d_i - 1} \ , \tag{5.5}$$

where the product includes (where appropriate) $i = 0$. As above, we assume the simplest normalization. In a "geometric" phase, this leads to the following situation. If $d_0 = 0$, we have $\chi^+ = \chi$. Moreover, the δ_0 insertion in (5.3) is absorbed in restricting to $\mathcal{M}_{\vec{n}}$ so that

$$Z^{\vec{n}}_{a_1 \ldots a_s} = \langle (\eta_{a_1})_{\vec{n}} \cdots (\eta_{a_s})_{\vec{n}} \chi_{\vec{n}} \rangle_{\vec{n}} \ . \tag{5.6}$$

On the other hand, for $d_0 < 0$, we have $\chi^+ = \xi_0^{-d_0-1}\chi$ and the insertion is not used to perform the restriction so that

$$Z^{\vec{n}}_{a_1\ldots a_s} = \langle(\eta_{a_1})_{\vec{n}}\cdots(\eta_{a_s})_{\vec{n}}\chi_{\vec{n}}(\xi_0)_{\vec{n}}^{-d_0}\rangle_{\vec{n}} \ . \tag{5.7}$$

Comparing these with (4.21) we see that for any monomial $\mathcal{O}$ in the σ_a

$$\langle\!\langle\mathcal{O}\rangle\!\rangle = \langle\!\langle\!\langle(-\delta_0^2)\mathcal{O}\rangle\!\rangle\!\rangle \ . \tag{5.8}$$

Thus the V^+ model, modified by the insertion of $(-\delta_0^2)$, computes the correlators of the M model.

It may thus seem that we have done little more than find a rather clumsy way to rewrite what we already knew, but we will see that this expression has some advantages over the previous one. For now, we note that if we define the quantum generalization of (5.1)

$$\mathrm{Tr}(\mathcal{O}) = \langle\!\langle\!\langle\delta_0\mathcal{O}\rangle\!\rangle\!\rangle \ , \tag{5.9}$$

then (5.8) in conjunction with the results of section four shows that the ring

$$\mathcal{R}_0(V) \equiv \mathbb{C}[\delta_0,\ldots,\delta_n]/\mathcal{J}_0 \ , \tag{5.10}$$

where

$$\mathcal{J}_0 \equiv \{\mathcal{P}\in\mathbb{C}[\delta_0,\ldots,\delta_n] \mid \mathrm{Tr}(\mathcal{O}\mathcal{P}) = 0 \quad \forall\mathcal{O}\} \ , \tag{5.11}$$

is a graded ring of length four. This ring will play a part in the argument of subsection *5.4*.

The relations in $\mathcal{J}_0$ include a set of linear relations, following from (3.51) and (5.2), which together can be written as

$$\sum_{i=0}^{n}\langle m^+, v_i^+\rangle\delta_i = 0 \ . \tag{5.12}$$

We find additional nonlinear relations by repeating the argument of subsection *3.9*. The considerations surrounding (3.80) express properties of the $\mathcal{M}_{\vec{n}}$. In the new model their interpretation (3.83) is modified, because instanton contributions are associated to factors of δ_0. Thus the modified relations read

$$\prod_{i=1}^{n}\delta_i^{d_i^*} = \prod_{a=1}^{n-d} q_a^{n_a^*}\delta_0^{-d_0^*} \ . \tag{5.13}$$

The modification renders the relations homogeneous as one expects for a graded ring. When V is smooth the arguments of Batyrev as presented in subsection *3.9* guarantee that these relations generate $\mathcal{J}_0$. (The key point is the fact – noted above – that in the limit as $q_a \to 0$ the ring $\mathcal{R}_0(V)$ approaches the classical cohomology ring $H^*(V)$.) As illustrated in appendix B, there can be additional relations when V fails to be smooth.

In terms of $\mathcal{R}_0(V)$ we can rewrite (5.8) as

$$\langle\langle \mathcal{O} \rangle\rangle = \mathrm{Tr}(-\delta_0 \mathcal{O}) \ . \tag{5.14}$$

We will think of this as determining the chiral ring of the **A** model with target space M (i.e., the quantum cohomology ring of M) in terms of $\mathcal{R}_0(V)$ as being

$$\mathcal{R}_A(M) = \mathcal{R}_0(V)/\{\mathcal{P} \mid \langle\langle \mathcal{O}\mathcal{P} \rangle\rangle = 0 \ \forall \mathcal{O}\} \ . \tag{5.15}$$

Notice that the form of (5.14) is again precisely what one expects from Nakayama's theorem – it defines $\langle\langle \ \rangle\rangle$ as an expectation function on a quotient algebra of $\mathcal{R}_0(V)$.

5.3. Other Phases

As discussed in section four, the correlation functions in the M model are rational functions of the parameters in the coordinates and frame given by our simple choices. This means their only singularities are poles, and analytic continuation into the complement of the singular locus in r-space is unambiguous. This continuation leads to predictions for the correlation functions in "nongeometric" phases. The analysis to this point has been restricted to the "geometric" phases, where we could observe that analytically continuing around a singular locus from one "geometric" phase to another leads to correlators which agree precisely with those computed in the new phase directly (a local argument showing how this happens is given in [6]). It is therefore very natural to assume that the same procedure will yield the correlation functions in the "nongeometric" phases as well, since the correlators must be holomorphic objects away from the singularities. Explicit computations in "nongeometric" phases can be difficult, though, and thus direct verification of this assumption has been unavailable. (Of course, in addition to the general argument of holomorphicity one can use mirror symmetry to prove this indirectly.)

One advantage of the symmetric treatment of ϕ_0 is that computations in other regions of r-space are now formally as simple as those for the "geometric" phases. The moduli spaces are determined from (3.54) in the same way; the Euler classes χ follow from (5.5).

The correlation functions are now computed as expansions about various limiting points in q corresponding to the various semiclassical limits in the model. These expansions will agree with each other (after analytic continuation). We will encounter a new subtlety in studying the "nongeometric" phases. In some instanton sectors we find an unbroken discrete subgroups $H \subset G$. As discussed in section four, when this occurs we should quotient by H. In practice this means that our naïve computation of the contribution to the path integral from these sectors will overcount by a factor of $|H|$. We will illustrate this in our two examples, performing the calculations in all non-smooth phases. (The orbifold phase of example 2 could have been discussed in the previous sections, but we treat it with the other phases now.)

Example 1.

We now consider the first example, in the Landau–Ginzburg phase $r \ll 0$. In this region we find $\mathcal{K}^\vee = \{n \le 0\}$ and the expansion is in "anti-instantons". A physical description of these was given in [6]. The instantons appear as Nielsen-Olesen vortices. Since the nonzero expectation value of ϕ_0 breaks $G = U(1)$ to $\mathbb{Z}_5$, we expect the instanton number to be quantized in $\frac{1}{5}\mathbb{Z}$. This would lead to an expansion of correlation functions in powers of $q^{-1/5}$. Analytically continuing a rational function, however, will never lead to branch singularities, so we will find, in fact, contributions only from integer instanton number. First we will need a description of the moduli spaces. For $n = 0$ the D term will require $\phi_0 \neq 0$ because of the sign of r. This however will never intersect our chosen representative of δ_0, hence we get no contribution from the zero sector. This is a general feature of "nongeometric" phases. For $n < 0$ we see that $d_i = n < 0$ so $\phi_i = 0$, whereas $d_0 = -5n > 0$. Here $Y_n = \mathbb{C}^{1-5n}$; as before $F_n = \{0\}$ is the origin. Thus $\mathcal{M}_{\vec{n}} = \mathbb{P}^{-5n}$. We also need $\chi_n = \prod_{i=1}^{5}(\xi_i)^{-d_i-1} = (-\xi_0/5)^{-5-5n}$. The unbroken $\mathbb{Z}_5$ symmetry in each sector means we need to divide our naïve result by 5.

We are interested in X_3 of (4.22)

$$X_3 = \langle\!\langle \sigma^3 \rangle\!\rangle = \langle\!\langle\!\langle (-\delta_0^2)\sigma^3 \rangle\!\rangle\!\rangle \ . \tag{5.16}$$

We will of course reexpress $\sigma = -\delta_0/5$ (and $\eta = -\xi_0/5$), and find

$$X_3 = \sum_{n<0} \langle (-5)^{5n+1}(\xi_0)^{-5n} \rangle_n q^n = \frac{5}{1+5^5 q} \ , \tag{5.17}$$

in agreement with (4.24) as expected.

Example 2.

Here there are, in addition to the smooth phase discussed in section four, three additional phases. We treat these in order. To compute the contributions of the various instanton numbers we will use

$$X_j^{\vec{n}} = \langle \eta_1^{3-j} \eta_2^j \chi_{\vec{n}}^+ \rangle_{\vec{n}} \ . \tag{5.18}$$

In phase II we find $\mathcal{K}^\vee = \{n_1 \geq 0,\ n_1 \geq 2n_2\}$. As in phase I, there are two regions to study. The region $n_1, n_2 \geq 0,\ n_1 \geq 2n_2$ comprises precisely those instanton configurations studied in the discussion of phase I which still make sense in the unresolved model. The contributions of these to correlation functions will be identical to those computed in section four by the arguments of subsection *5.2* (except for the contribution of $\vec{n} = 0$ – nontrivial since we are in a "geometric" phase – which we include separately by hand). The new region is $n_1 \geq 0,\ n_2 < 0$. In this region $d_1 = d_2 < 0$; setting $\phi_1 = \phi_2 = 0$ the moduli space is seen to be $\mathbb{P}^{3n_1+2} \times \mathbb{P}^{n_1-2n_2}$ with the hyperplane section of the first factor corresponding to η_1 and of the second to ξ_6. The unbroken discrete subgroup is $\mathbb{Z}_2 \subset G_2$. Here $\chi_{\vec{n}}^+ = \eta_2^{-2-2n_2}(4\eta_1)^{4n_1+1}$ hence

$$X_j^{\vec{n}} = 2^{8n_1+2n_2+3-j} \binom{j-2-2n_2}{n_1-2n_2} \ . \tag{5.19}$$

This is nonzero for $n_1 \leq j-2$, and agrees precisely with (4.28).

In phase III we have $\mathcal{K}^\vee = \{n_1 \leq 0,\ n_1 \geq 2n_2\}$. This will lead to the simplest computation of the correlation functions, a general property of Landau–Ginzburg phases. In this (nongeometric) phase we see that $\vec{n} = 0$ will not contribute – once more a constant ϕ_0 would need to be nonzero to solve $D = 0$. Here $d_i < 0$ for $1 \leq i \leq 5$, and the moduli space is $\mathcal{M}_{\vec{n}} = \mathbb{P}^{-4n_1} \times \mathbb{P}^{n_1-2n_2}$; the hyperplane classes are $\xi_0,\ \xi_6$ respectively. Also $\chi_{\vec{n}}^+ = \eta_2^{-2-2n_2}\eta_1^{-3-3n_1}$, and $H = \mathbb{Z}_2 \times \mathbb{Z}_4$. We find

$$X_j^{\vec{n}} = -2^{8n_1+2n_2+3-j} \binom{j-2-2n_2}{j-2-n_1} \ . \tag{5.20}$$

In phase IV, $\mathcal{K}^\vee = \{n_1 \leq 0,\ n_2 \geq 0\}$; once more $n_1 = 0$ cannot contribute. Here we have $\mathcal{M}_{\vec{n}} = \mathbb{P}^{-4n_1} \times \mathbb{P}^{2n_2+1}$, with hyperplane classes ξ_0 and η_2, and $\chi_{\vec{n}}^+ = \eta_1^{-3-3n_1}\xi_6^{2n_2-n_1-1}$, and $H = \mathbb{Z}_4 \subset G_1$. Hence

$$X_j^{\vec{n}} = 2^{8n_1+2n_2+3-j} \binom{2n_2-n_1-1}{j-2-n_1} \ . \tag{5.21}$$

We see that the expansions in all phases agree as expected. The enclosed region in figure 3 is interesting. It is inherently strongly-coupled in the sense that none of the instanton expansions about semiclassical limiting points converges there. Thus we do not have an understanding of the low-energy degrees of freedom in this region even in the limited sense in which this exists for, say, phase IV.

5.4. Mirror Symmetry

We now turn to Batyrev's construction of a (conjectured) mirror manifold to M. This is realized in a toric variety Λ determined by combinatorial data which are dual in a natural sense to those of V. To prove the conjecture we would need to show that the superconformal field theories determined by the two manifolds are isomorphic. In this paper we will do quite a bit less. What we can attempt with our methods is to prove that the correlation functions we are able to compute in the **A** model – those of the operators which are related to cohomology classes in $H^*_V(M)$ – are equal to suitably defined correlators in the **B** model on W. The mirror counterpart of $H^*_V(M)$ will turn out to be the subring of $\oplus H^p(\wedge^q TW)$ which is generated by polynomial deformations of the complex structure. We will denote this subring by $P_\Lambda(W)$. In proving the isomorphism we will also find a mirror map for the "toric" parameter spaces of the two theories – the spaces of deformations generated by the elements in the ring at degree one. We note that while falling short of a complete proof of mirror symmetry, this is a much stronger statement than what has previously been proved, namely the equality of respective Hodge numbers [26] and of the asymptotic ($q = 0$) limits of (some of) the correlators [30,8,65]. We will manage to establish the equality of correlation functions under the following hypotheses: (i) M is realized as a hypersurface in a *smooth* toric variety V. This will allow us to use the algebraic solution of subsection *3.9* or, equivalently, (5.13). (ii) The fan ∇ determining the model for Λ in which the canonical divisor is ample is *simplicial*. This technical requirement means that the cones in the fan are all cones over simplices. We will need this condition because various properties of the **B** model on W that we will use are proved (at present) only when it holds.

The method of proof will be the following. We will construct from the data of the **B** model on W a ring $P_0(\Lambda)$ which we call the *extended chiral ring*. This ring – which has been studied by Batyrev [66,8] in a slightly different formulation – is related to the

mixed Hodge structure on the affine hypersurface $W \cap \mathcal{T}^\vee \subset \mathcal{T}^\vee$, where[22] $\mathcal{T}^\vee \sim (\mathbb{C}^*)^d$ is the torus contained in the toric variety Λ. We will construct a natural isomorphism – the *global monomial–divisor mirror map* – between $P_0(\Lambda)$ and the ring $\mathcal{R}_0(V)$ of (5.10).

The ring $\mathcal{R}_0(V)$ comes with an expectation function (5.9). The isomorphism we establish will allow us to use this as an expectation function on $P_0(\Lambda)$. Further, under condition (ii) above, Batyrev and Cox [67] and Cox [68] have shown that the ring $P_0(\Lambda)$ is related to the chiral ring of the **B** model on W in the following way. The subring of the chiral ring which is generated by polynomial deformations of complex structure will be denoted by $\mathcal{R}_B(W)$ and referred to as the *restricted chiral ring.* This ring $\mathcal{R}_B(W)$ is the quotient of $P_0(\Lambda)$ by an ideal, with the construction corresponding precisely to (5.14) and (5.15), in terms of a certain expectation function on $P_0(\Lambda)$ defined by Batyrev and Cox and a certain distinguished element in the ring (the analogue of δ_0). We then appeal to the arguments given in section two regarding expectation functions on graded rings to claim that the two expectation functions on $P_0(\Lambda)$ are in fact identical (up to a normalization), hence finally that the specified subrings of the chiral rings of the two models agree.

The first relationship of this kind was found by Batyrev [66,8], who related a part of the Hodge structure of the affine hypersurface to the quantum cohomology of the ambient space of the mirror. Our isomorphism is closely related to his.

Let us begin by noting some facts about the restricted chiral ring $\mathcal{R}_B(W)$. We recall that W is a hypersurface in the dual toric variety Λ described in subsection *5.1.* The homogeneous coordinates $y_1, \ldots, y_u$ of Λ correspond to the vertices of the polytope $\mathcal{P}^0$, since we will not include in the fan ∇ any other lattice points. This choice ensures that the canonical class of Λ is an ample divisor class. In terms of these, the hypersurface W is determined by an equation $f(y) = 0$, with f a polynomial of appropriate multidegree. We write f in the form

$$f(y) = \sum_{i=0}^{n} c_i \mu_i \ , \tag{5.22}$$

for some coefficients c_i, where μ_i runs over all the monomials in y of the correct multidegree. The coefficients c_i parameterize the "toric part" of the deformation space, and can be regarded as homogeneous coordinates on that space. The $\mathbb{C}^{*u}$ action on the original homogeneous coordinates y_j induces an $\mathbb{C}^{*d}$ action on Λ, hence on the coefficients c_i. The

[22] The notation $\mathcal{T}^\vee$ indicates that this torus is dual to the torus $\mathcal{T}$ which is contained in the toric variety V.

"toric" deformation space is a toric variety of dimension $n-d$ which can be described as a quotient of the space of coefficients by this $\mathbb{C}^{*d}$ action (see [30] and [39] for more details).

Under condition (ii), Batyrev and Cox [67] show that the restricted chiral ring $\mathcal{R}_B(W)$ can be described as follows. Start with the "affine Jacobian ideal" of f, which is the ideal $\mathcal{J}_0(f)$ generated by the "affine" partial derivatives $y_j \frac{\partial f}{\partial y_j}$, and define $\mathcal{J}_1(f)$ to be the "ideal quotient" of $\mathcal{J}_0(f)$ by the monomial $\mu_0 := \prod y_j$, that is,

$$\mathcal{J}_1(f) = \mathcal{J}_0(f){:}\mu_0 = \{\mathcal{P} \mid \mu_0\mathcal{P} \in \mathcal{J}_0(f)\} \ . \tag{5.23}$$

(In the case of a weighted projective space, this ideal agrees with the ordinary Jacobian ideal $\mathcal{J}(f)$ – the one generated by the partial derivatives $\frac{\partial f}{\partial y_j}$ – but in general $\mathcal{J}_1(f)$ is larger than $\mathcal{J}(f)$.) Then $\mathcal{R}_B(W)$ is the subalgebra of

$$\mathcal{R}_1(f) = \mathbb{C}[y_1, \ldots, y_u]/\mathcal{J}_1(f) \ , \tag{5.24}$$

generated by all elements whose multidegree is the same as that of f.[23] The algebra $\mathcal{R}_B(W)$ has natural monomial generators which are in one-to-one correspondence with the holomorphic sections of the anticanonical line bundle of Λ, i.e., with the points in $\mathcal{P}$. Condition (i) then means these are precisely the generators of the one-dimensional cones in Δ,[24] as well as the unique interior point of $\mathcal{P}$. To each cone, in turn, we associated one of the homogeneous coordinates x_i on V, and the corresponding divisor ξ_i. Let us make this correspondence more concrete. The generators of one-dimensional cones in Δ were denoted $v_1, \ldots, v_n$. Let $v_0 = 0$, and let v_i^+ denote the "promoted" v_i, a vector in $\mathbf{N}_{\mathbb{R}}^+$ whose last component is 1. Similarly, let $\lambda_1, \ldots, \lambda_u$ be the generators of one-dimensional cones in ∇, and λ_0, λ_j^+ as above. Then to v_i we associate the monomial

$$\mu_i = \mu_i(y) = \prod_{j=1}^{u} y_j^{\langle \lambda_j^+, v_i^+ \rangle} = \prod_{j=1}^{u} y_j^{1+\langle \lambda_j, v_i \rangle} \ , \tag{5.25}$$

[23] Batyrev and Cox actually show that the primitive cohomology of the hypersurface is isomorphic to the subalgebra of $\mathcal{R}_1(f)$ which consists of all elements whose multidegree is a *multiple* of that of f. Our "restricted chiral algebra" is the subalgebra of this generated by elements of multidegree precisely $|f|$; when the hypersurface is a threefold this coincides with the primitive cohomology.

[24] We ignore the subtlety associated to points in the interior of codimension-one faces. This can be incorporated at the price of a somewhat more cumbersome discussion.

such that $y_0\mu_i$ is a possible term in the superpotential for the W model. It is convenient to include the coefficient c_i along with the monomial μ_i, and to present the algebra $\mathcal{R}_B(W)$ as

$$\mathcal{R}_B(W) = \mathbb{C}[c_0\mu_0, \ldots, c_n\mu_n]/\mathcal{J}_1'(f) \ , \tag{5.26}$$

where $\mathcal{J}_1'(f)$ is the inverse image of $\mathcal{J}_1(f)$ under the map

$$\mathbb{C}[c_0\mu_0, \ldots, c_n\mu_n] \to \mathbb{C}[y_1, \ldots, y_u] \tag{5.27}$$

induced by (5.25).

Among the relations in $\mathcal{J}_1'(f)$ are those in the kernel of (5.27), which follow simply from the expression (5.25) for the μ_i. These are generated by relations corresponding to sets $n_1^*, \ldots, n_{n-d}^*$ such that $d_i^* = \sum_{a=1}^{n-d} Q_i^a n_a^* \geq 0 \ \forall i > 0$, given by

$$\prod_{i=1}^{n} {\mu_i}^{d_i^*} = {\mu_0}^{\Sigma d_i^*} \ , \tag{5.28}$$

or, written in terms of the generators of $\mathbb{C}[c_0\mu_0, \ldots, c_n\mu_n]$,

$$\prod_{i=1}^{n} (c_i\mu_i)^{d_i^*} = (c_0\mu_0)^{\Sigma d_i^*} \prod_{i=1}^{n} \left(\frac{c_i}{c_0}\right)^{d_i^*} \ . \tag{5.29}$$

Further relations in $\mathcal{J}_1'(f)$ come from the "affine" derivatives $y_j \frac{\partial f}{\partial y_j}$. These are of degree $|f|$ and can thus be themselves written as linear combinations of the μ_i's, leading to *linear* relations among the latter. There are u such relations, but the homogeneity of f implies the existence of Euler equations relating these, so that precisely $d{+}1$ are independent. It is convenient to write these relations in terms of the one-parameter group actions on Λ^+. Given $m^+ \in \mathbf{M}^+$, (5.25) and (5.22) yield as the linear relations in $\mathcal{J}_1'(f)$ the following:

$$\sum_{i=0}^{n} \langle m^+, v_i^+ \rangle \, c_i\mu_i = 0, \quad p = 1, \ldots, d \ . \tag{5.30}$$

Now we see why the coefficients c_i were included with the generators – doing so gives the linear relations (5.30) a universal form, independent of the choice of coefficients.

The relations derived in the previous paragraph do not generate the entire ideal $\mathcal{J}_1'(f)$ (because we have used the "affine" derivatives rather than $\frac{\partial f}{\partial y_j}$, and because we have not yet included the "extra" elements which enlarge $\mathcal{J}_1(f)$ beyond the Jacobian ring $\mathcal{J}(f)$). We let $\mathcal{J}_0(f) \subset \mathbb{C}[y_1, \ldots, y_u]$ be the ideal generated by the affine derivatives $y_j \frac{\partial f}{\partial y_j}$ (note

that $\mathcal{J}_0(f) \subset \mathcal{J}(f) \subset \mathcal{J}_1(f)$), and let the *extended chiral ring* $P_0(\Lambda)$ be the subalgebra of $\mathbb{C}[y_1, \ldots, y_u]/\mathcal{J}_0(f)$ generated by all elements whose degree is divisible by $|f|$. This extended chiral ring can be presented as

$$P_0(\Lambda) = \mathbb{C}[c_0\mu_0, \ldots, c_n\mu_n]/\mathcal{J}_0'(f) \ , \tag{5.31}$$

where $\mathcal{J}_0'(f)$, the inverse image of $\mathcal{J}_0(f)$ under (5.27), is generated by (5.29) and (5.30). According to [67], this ring is isomorphic to the mixed Hodge structure on the dth primitive cohomology of the affine hypersurface $W \cap T^\vee$ (which had been studied by Batyrev [66] using a different presentation).

The isomorphism between $\mathcal{R}_0(V)$ and $P_0(\Lambda)$ is given by the simple mapping

$$\begin{aligned} \delta_i &\leftrightarrow c_i\mu_i \\ q_a &\leftrightarrow \prod_{i=1}^{n} (c_i/c_0)^{Q_i^a} \ , \end{aligned} \tag{5.32}$$

under which the relations (5.13) are mapped to (5.29), and the relations (3.8) and (5.2) are mapped to (5.30). This equation is precisely the *monomial-divisor mirror map* (up to sign[25]) of [39], and is closely related to maps appearing in [8]. In those works only the asymptotic behavior of the map was studied in the **A** model; we see here that the q_a give a precise meaning to this map throughout parameter space. For this reason, we refer to (5.32) as the *global* monomial-divisor mirror map.

The fact that the relations for $\mathcal{R}_0(V)$ – which were derived from studying instanton moduli spaces and quantum cohomology – are exactly the same as those for $P_0(\Lambda)$ – derived from the Jacobian ring of a polynomial – is quite remarkable, in fact, completely unexpected from a mathematical point of view. There is an additional remarkable correspondence. As we have seen in subsections *3.5* and *4.2*, the set of q_a's for which the V^+ model is singular has a very explicit description, given by (4.11) and (4.12). On the **B** model side there is an equally explicit description of the singularities [69]. The hypersurface with equation $\sum c_i\mu_i = 0$ fails to be quasi-smooth precisely when $E_A(c_i) = 0$, where E_A is a polynomial called the *principal A-determinant* of the set $A := \{\lambda_0, \lambda_j^+\}$. Each irreducible component of

[25] As we have discussed earlier, although our comparison between **A** and **B** model correlation functions involves no signs, some signs are needed in order to correctly compare linear and non-linear σ-models. We have given a formula for these in (4.20). This differs from a conjectured form of the signs given in [60], based upon ideas from [69].

the set $E_A = 0$ is associated to a face Γ of $\mathcal{P}$ (including the "face" $\Gamma = \mathcal{P}$), and coincides with the zero-locus of another polynomial called the *A-discriminant* $\Delta_{A\cap\Gamma}$. We saw a similar structure on the **A** model side in subsection *3.5*, in which faces of the polytope $\mathcal{P}$ were associated to components of the singular locus.

In addition, the structure of the components $\Delta_{A\cap\Gamma} = 0$ is known in detail. The main idea goes back to a 19th century paper of Horn [70], but it has been put into modern form by Kapranov [71]: taking as coordinates on the parameter space the expressions $p_a := \prod_{i=1}^n (c_i/c_0)^{Q_i^a}$ (which correspond to q_a under the monomial-divisor mirror map), the zeros of the principal discriminant $\Delta_A(p_a)$ are precisely the p_a's for which

$$p_a = \prod_i \left(\sum_{b=1}^{n-d} Q_i^b \ell_b \right)^{Q_i^a} \tag{5.33}$$

for some values of ℓ_b. The other components (with defining polynomials $\Delta_{A\cap\Gamma}(p_a)$) admit a similar description: in appropriate coordinates, they are the p_a's for which

$$p_a = \prod_{i\in I} \left(\sum_{b=n-d-k+1}^{n-d} Q_i^b \ell_b \right)^{Q_i^a} \quad , \ a = n{-}d{-}k{+}1, \ldots, n{-}d \ , \tag{5.34}$$

for some values of ℓ_b, where $\{\mu_i\}_{i\in I}$ is the set of monomials in the face Γ of $\mathcal{P}$, where $Q_i^a = 0$ for all $i \notin I$ and $a > n{-}d{-}k$, and where $\vec{Q}^{n-d-k+1}, \ldots, \vec{Q}^{n-d}$ span the set of all $\vec{Q}$'s for which $\vec{Q}_i = 0 \ \forall \ i \notin I$. But this exactly corresponds to the description of the components of the singular locus we found on the **A** model side!

We turn now to the expectation functions on the algebra $\mathcal{R}_0(V) \cong P_0(\Lambda)$. This is a graded Frobenius algebra, and as discussed in section two, has a graded expectation function which is unique up to scalar multiple. Each of our descriptions of this algebra comes equipped with an expectation function: the "trace" map Tr given by (5.9) defines a graded expectation function on $\mathcal{R}_0(V)$, and the "toric residue" Res of [68] defines a graded expectation function on $P_0(\Lambda)$. The scalar multiple which relates these may in principle depend on q, so using the isomorphism (5.32) we have

$$\mathrm{Tr}(\mathcal{O}) = s(q)\,\mathrm{Res}(\mathcal{O}) \ , \tag{5.35}$$

for some scaling function $s(q)$.

Under the global monomial-divisor mirror map (5.32), the special class δ_0 is mapped to the special monomial $c_0\mu_0 = c_0 \prod y_j$. The quantum cohomology ring of M is determined

by insertion of $(-\delta_0)$ as described in (5.14); remarkably, when Λ is simplicial it follows from results in [67,68] that the expectation function $\langle\!\langle\ \rangle\!\rangle_B$ on the restricted chiral ring of W is given by[26]

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_B = \operatorname{Res}\left((-c_0\mu_0)\mathcal{O}\right) \ . \tag{5.36}$$

In fact, as mentioned earlier the restricted chiral ring of W is described in [67,68] as the subalgebra of $\mathbb{C}[y_1, \ldots, y_u]/\mathcal{J}_1(f)$ generated by all elements whose degree is divisible by $|f|$, where $\mathcal{J}_1(f)$ is the ideal quotient $\mathcal{J}_0(f){:}\mu_0$ defined in (5.23). When $c_0 \neq 0$ this is clearly the same as modding out by the annihilator of $c_0\mu_0$, as we expect from Nakayama's theorem.

Comparing (5.36) to (5.14) we conclude that under the monomial-divisor mirror map we have

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_A = s(q)\langle\!\langle \mathcal{O} \rangle\!\rangle_B \ , \tag{5.37}$$

where the subscript on the left-hand side is added for clarity.

Mirror symmetry naïvely predicts $s(q) = 1$. But of course this ignores the fact that correlation functions are sections of a line bundle over parameter space and thus defined as functions only after a choice of trivialization. In this sense (5.37) is a proof of mirror symmetry and the mirror map used (even if $s(q)$ is nontrivial). In fact, we claim that the mirror map can be extended to mirror choices of trivializations as well; in other words, we can give an interpretation in the context of the **B** model of the gauge choice we made implicitly in the **A** model. The gauge in question is what we will call *algebraic gauge* and is defined (in part) by the property that correlation functions be meromorphic functions. (This is not possible in arbitrary coordinate systems but is manifestly possible in the q coordinates.) This property restricts the unknown function $s(q)$ to being a meromorphic function. We can use the compactification of parameter space as a toric variety [39] to claim that such a function should be determined by its singularities. Moreover, we have found the exact location for the singularities on both the **A** and **B** model sides, and they correspond under our map. This means that $s(q)$ is a meromorphic function whose zeros and poles are all located along the components of the singular locus (since a zero or pole elsewhere would cause a mismatch between singularity sets of the two theories). It seems likely that $s(q)$ must in fact be a constant, but we have no general proof of this. In any case,

[26] The rôle of the "fundamental monomial" $c_0\mu_0$ in understanding the chiral ring of a hypersurface was first considered by Hübsch and Yau [72].

though, the monomial-divisor mirror map determines a natural gauge on the **B** model side, which corresponds to the **A** model's algebraic gauge under the monomial-divisor mirror map.

Note that our hypotheses (i) and (ii) have been used in the following way: (i) allowed us to give a relatively simple algebraic description of the ring $\mathcal{R}_0(V)$ and the correlation function $\langle\!\langle\ \rangle\!\rangle_A$, and (ii) allowed us to give a similar a description of the extended chiral ring $P_0(\mathit{\Lambda})$ and the correlation function $\langle\!\langle\ \rangle\!\rangle_B$. We would need extensions of these descriptions on both sides in order to verify the isomorphism in a wider context.

There is a class of examples in which we can make the preceding discussion much more concrete. This is the case in which the polynomial f, considered as a function $\mathbb{C}^u \to \mathbb{C}$, can be chosen to be *nonsingular* away from the origin in $\mathbb{C}^u$. This condition must hold, for example, if the W model has a Landau–Ginzburg phase. In this case the toric residue Res coincides with the Grothendieck residue formula, and provides a natural expectation function on $\mathcal{R}_B(W)$ of the form

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_B = \oint \prod_j dy_j \, \frac{\mathcal{O}(y)}{\prod_j \frac{\partial f}{\partial y_j}} \ . \tag{5.38}$$

This is manifestly in algebraic gauge. Given the explicit formula (5.38) we see that this is obtained by an application of (5.35) (with $s(q)$ constant) from an expectation function on $P_0(\mathit{\Lambda})$ given by

$$\mathrm{Tr}(\mathcal{O}) = -\frac{1}{c_0} \oint \prod_j dy_j \, \frac{\mathcal{O}(y)}{\prod_j y_j \frac{\partial f}{\partial y_j}} \ . \tag{5.39}$$

6. Open Problems

The results in this paper are an application of the proposal of [6] that the nonlinear sigma model for target spaces related to toric varieties can be studied with the aid of the massive Abelian gauge theory which reduces to it at extremely low energies. We have seen that for the twisted **A** model with target space V a toric variety or $M \subset V$ a Calabi–Yau hypersurface these ideas suffice to lead to an essentially complete solution. The solution is unsatisfactory only in being naturally obtained in a set of coordinates which, in the hypersurface case, differ from the canonical set. To complete the picture one should compute the required change of coordinates using the methods described. Essentially, the coordinate change is encoded in the combinatorics of the relation between the compactified moduli spaces of instantons used here and the actual mapping spaces obtained in the

nonlinear model. Since this relation is explicitly known it should be possible to extract the coordinate change.

At a technical level this work raises quite a few questions. The two formulations of sections four and five yield two expressions for the quantum cohomology of M in terms of the quantum cohomology of V. Consistency of the construction requires agreement of these two formulas, but we have not explicitly showed this. The generator-relation presentation of the quantum cohomology algebra for a smooth V should have a generalization to a general toric variety. It appears the relations are not integrable into a twisted superpotential as in the smooth case. Integrating them would thus require the introduction of auxiliary fields (in addition to σ). It would be of interest to characterize these and perhaps their physical interpretation.

We have obtained useful results for the case of Calabi–Yau hypersurfaces in toric varieties. The GLSM can describe in the deep infrared nonlinear models on arbitrary complete intersections of such hypersurfaces, and it should be a simple matter to extend our results to this case. In particular, the description of the instanton moduli spaces should be no harder to obtain. We note, however, that the arguments leading to the quantum restriction formula would not hold in this case; presumably the validity of this is restricted to hypersurfaces. We see no such obstruction, however, to applying the methods of section five. Similarly, we have restricted our attention to Calabi–Yau subvarieties because these are the ones relevant for string theory. The methods of section four however should lead to an explicit computation of the quantum cohomology of *any* hypersurface in a toric variety.

The massive GLSM was crucial for obtaining all of the explicit results on which this work is based. It is however not clear how many of the qualitative conclusions depend upon the relation to toric varieties. For example, the arguments leading to (4.31) could probably be reproduced for a Calabi–Yau hypersurface M in an arbitrary Fano variety X, leading to a computation of the quantum cohomology of M in terms of the quantum cohomology of X. It would be interesting to make this construction explicitly, as well as to verify to what extent the same holds for the methods of section five. One can imagine adding a superpotential interaction to the action for the **A** twisted model with target space X^+ the total space of the canonical line bundle over X. This could be chosen so that the space of classical vacua reduces to $M \subset X$.

Beyond the ability to solve particular examples the interest in mirror symmetry arises from the hope that it reflects a general property of string theory. Our current state of understanding of the phenomenon – we know a few examples of mirror pairs – does not

allow us to address this question at all. To this end the results of the present work are at best a small step. A more meaningful one would be a manifestly mirror-symmetric construction of the models. The description of the dynamics of Σ_a directly in terms of the twisted superpotential is a promising hint in this direction, which we will pursue further in [7]. Of course, this is at present limited to models related to toric varieties.

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Appendix A. Some Details Concerning Example 2

We want to compute

$$Y_j^{(n)} = 4^{4n_1+1} \langle y^j z^{4n_1+4-j} \rangle_{\mathcal{M}_n} \tag{A.1}$$

using the recursion relations (3.74) that follow from

$$\begin{aligned} y_1^{n_2+1} y_2^{n_2+1} &= y^{2n_2+2} = 0 \\ y_6^{n_1-2n_2+1} y_3^{n_1+1} y_4^{n_1+1} y_5^{n_1+1} &= z^{3(n_1+1)} (z-2y)^{n_1-2n_2+1} = 0 \ . \end{aligned} \tag{A.2}$$

So denote as always

$$\varphi_a = 2^{-a} \langle y^{2n_2+1-a} z^{4n_1-2n_2+3-a} \rangle_{\mathcal{M}_n} \tag{A.3}$$

with $\varphi_a = 0$ for $a < 0$ and $\varphi_0 = 1$. Then (A.2) implies

$$2^{-a} \varphi_a + \sum_{j=1}^{n_1-2n_2+1} (-2)^j \binom{n_1-2n_2+1}{j} 2^{a-j} \varphi_{a-j} = 0 \tag{A.4}$$

for $1 \leq a \leq 2n_2+1$. These relations determine the φ_a completely. We claim this is solved by

$$\varphi_a = \binom{n_1 - 2n_2 + a}{a} \,. \tag{A.5}$$

The proof is trivial. We need to show

$$\binom{m+a}{a} + \sum_{j=1}^{m+1} (-1)^j \binom{m+1}{j} \binom{m+a-j}{a-j} = 0. \tag{A.6}$$

Note that (a) we can incorporate the first term into the sum as $j=0$; (b) this is the same as what we need as indeed $\binom{m}{n}$ vanishes for $m>0,\ n<0$. With the (true) observation that $n!$ diverges for $n<0$ we rewrite this as

$$\frac{1}{m!} \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{(m+a-j)!}{(a-j)!} \tag{A.7}$$

where all negative factorials are to be understood as Gamma functions. However, if we let

$$F(x) = x^{m+a}(1-\frac{1}{x})^{m+1} = x^{a-1}(x-1)^{m+1} \tag{A.8}$$

then expanding yields

$$F(x) = \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} x^{m+a-j} \tag{A.9}$$

hence

$$\frac{d^m F}{dx^m} = \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{(m+a-j)!}{(a-j)!} x^{a-j} \tag{A.10}$$

(with same understanding as above). Setting $x=1$ we have a proof of (A.6).

Appendix B. A Singular Example

In this appendix we perform the instanton sum calculation in an example for which V is not smooth. Thus Batyrev's algebraic solution to the quantum cohomology problem is not valid. We show that the instanton computations are well-defined, and sum to the result predicted by mirror symmetry, in this case as well. The mirror manifold was studied in [46].

We start with $\mathbb{P}_4^{1,2,2,3,4}$ (in which a degree 12 hypersurface is Calabi–Yau). This has a codimension two set of $\mathbb{Z}_2$ singularities at $x_1 = x_3 = 0$. It also has codimension

four singularities at (0,0,0,1,0) ($\mathbb{Z}_3$) and (0,0,0,0,1) ($\mathbb{Z}_4$). The first we resolve easily; the resulting fan is spanned by

$$\begin{aligned}
v_1 &= (-2,-2,-3,-4) \\
v_2 &= (1,0,0,0) \\
v_3 &= (0,1,0,0) \\
v_4 &= (0,0,1,0) \\
v_5 &= (0,0,0,1) \\
v_6 &= (-1,-1,-1,-2) = 1/2(v_1+v_4)\ .
\end{aligned} \tag{B.1}$$

Find the kernel to compute the charges

$$Q = \begin{pmatrix} -1 & 1 & 1 & 0 & 2 & 3 & -6 \\ 1 & 0 & 0 & 1 & 0 & -2 & 0 \end{pmatrix} \tag{B.2}$$

(adding the seventh x_0 column).

Using this find the phase structure straightforwardly. There are five phases, as follows (we give cone in r space as well as the indices of the primitive collections of coordinate hyperplanes)

I CY, $r_1 > 0,\ r_2 > 0$; (14),(2356).

II $\mathbb{P}^4$, $r_2 < 0,\ 3r_2 + 2r_1 > 0$; (6),(12345).

III $\mathbb{P}^4$ with exoflop, $r_1 < 0,\ r_2 + r_1 > 0$; (10),(23456).

IV Hybrid, $r_2 + r_1 < 0,\ r_2 > 0$; (0),(14).

V LG, $r_2 < 0,\ 3r_2 + 2r_1 < 0$; (0),(6).

To see the failure of the algebraic solution we concentrate on the smooth Calabi–Yau phase; the classical relations are (from F)

$$\begin{aligned}
\xi_1\xi_4 &= 0 \\
\xi_2\xi_3\xi_5\xi_6 &= 0\ .
\end{aligned} \tag{B.3}$$

We have $d = (n_2 - n_1, n_1, n_1, n_2, 2n_1, 3n_1 - 2n_2, -6n_1)$ (again include d_0 last). The cone $\mathcal{K}^+$ of instantons such that all $d_i \geq 0$ is thus spanned by (1,1) and (2,3) which lead to the relations (eqn. (5.13))

$$\begin{aligned}
\delta_2\delta_3\delta_4\delta_5^2\delta_6 &= q_1 q_2 \delta_0^6 \\
\delta_1\delta_2^2\delta_3^2\delta_4^3\delta_5^4 &= q_1^2 q_2^3 \delta_0^{12}\ .
\end{aligned} \tag{B.4}$$

Dividing second by square of first indeed get $\delta_1\delta_4 = q_2\delta_6^2$ as expected. However, these do not suffice to obtain the quantum cohomology of M.

To compute the correlators we will work in the Landau–Ginzburg phase. This leads to the simplest computation and by analytic continuation determines the correlators in all phases. We have $\mathcal{K}^\vee_{\mathrm{LG}} = \{n_1 < 0,\ 3n_1 - 2n_2 > 0\}$ (the form of F here shows that we need the strong inequalities). From d we see that the moduli space is $\mathcal{M}_{\vec{n}} = \mathbb{P}^{3n_1-2n_2} \times \mathbb{P}^{-6n_1}$ with the hyperplane sections of the two factors given by ξ_6, ξ_0 respectively. The general correlation function is given by

$$\langle\!\langle \mathcal{O} \rangle\!\rangle = \sum_{\vec{n}\in\mathcal{K}^\vee} (-q_1)^{n_1} q_2^{n_2} \langle \mathcal{O}\xi_0^2 \xi_1^{n_1-n_2-1} \xi_2^{-n_1-1} \xi_3^{-n_1-1} \xi_4^{-n_2-1} \xi_5^{-2n_1-1} \rangle_{\vec{n}} \ . \tag{B.5}$$

Now rewrite everything in terms of ξ_0, ξ_6 using linear relations, and use $\mathcal{O}_j = \xi_0^{3-j}\xi_6^j$ as a basis, find

$$\langle\!\langle \mathcal{O}_j \rangle\!\rangle = 2^5 3^4 \sum_{\substack{m_1>0 \\ 2m_2>3m_1}} r_1^{m_1} r_2^{m_2} \langle \xi_0^{4m_1-j+2} \xi_6^j (\xi_0 + 2\xi_6)^{m_2-1} (\xi_0 + 6\xi_6)^{m_2-m_1-1} \rangle_{\vec{n}} \tag{B.6}$$

where $r_1 = \frac{1}{27q_1},\ r_2 = \frac{1}{48q_2}$. This now needs evaluation. From the expression for $\mathcal{M}_{\vec{n}}$ we can rewrite it as

$$\langle\!\langle \mathcal{O}_j \rangle\!\rangle = 2^5 3^4 \sum_{\substack{m_1>0 \\ 2m_2>3m_1}} r_1^{m_1} r_2^{m_2} \oint dx x^{3m_1-2m_2+j-1} (1+2x)^{m_2-1} (1+6x)^{m_2-m_1-1} \ . \tag{B.7}$$

Now exchange the sum with integral and sum the geometric series that appear to get

$$\langle\!\langle \mathcal{O}_j \rangle\!\rangle = 2^5 3^4 \oint dx\, x^{j-1} \frac{x^{-2} r_1^2 r_2^4 (1+2x)^3(1+6x) + x^{-1} r_2^2 (1+2x)}{[1 - r_2 x^{-2}(1+2x)(1+6x)]\,[1 - r_1^2 r_2^3 (1+2x)^3(1+6x)]} \tag{B.8}$$

where the two terms in numerator arise from odd and even m_1 respectively. For $j = 0$ the integrand has a pole at $x = 0$. For any j, however, there are poles at the roots of $x^2 = r_2(1+2x)(1+6x)$. These are of order r_2 and hence should be considered encircled by our contour. The correct Yukawas (which match the **B** model ones calculated in [46]) follow from simply adding up the residues about these poles.

References

[1] M. T. Grisaru, W. Siegel, and M. Roček, "Improved methods for supergraphs," Nucl. Phys. **B159** (1979) 429–450; for a recent version with a simpler proof see N. Seiberg, "Naturalness versus supersymmetric non-renormalization theorems," Phys. Lett. **318B** (1993) 469–475, hep-ph/9309335.

[2] See *Essays on mirror manifolds*, S.-T. Yau, ed. International Press, Hong Kong, 1992; *Essays on mirror manifolds II*, B. R. Greene and S.-T. Yau, eds. International Press, Hong Kong, to appear.

[3] E. Witten, "Topological sigma models," Commun. Math. Phys. **118** (1988) 411–449.

[4] E. Witten, "Mirror manifolds and topological field theory," in *Essays on Mirror Manifolds*, S.-T. Yau, ed. International Press, Hong Kong, 1992, pp. 120–159, hep-th/9112056.

[5] J. Distler and B. Greene, "Exact results on the superpotential from Calabi–Yau compactifications," Nucl. Phys. **B309** (1988) 295–316.

[6] E. Witten, "Phases of N=2 theories in two dimensions," Nucl. Phys. **B403** (1993) 159–222, hep-th/9301042.

[7] D. R. Morrison and M. R. Plesser, work in progress.

[8] V. V. Batyrev, "Quantum cohomology rings of toric manifolds," in *Journées de Géométrie Algébrique d'Orsay (Juillet 1992)*, Astérisque, vol. 218, Société Mathématique de France, 1993, pp. 9–34, alg-geom/9310004.

[9] P. S. Aspinwall and D. R. Morrison, "Chiral rings do not suffice: N=(2,2) theories with nonzero fundamental group," Phys. Lett. **334B** (1994) 79–86, hep-th/9406032.

[10] E. Witten, "The N-matrix model and gauged WZW models," Nucl. Phys. **B371** (1992) 191–245.

[11] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, 1962.

[12] G. Karpilovsky, *Symmetric and G-algebras: With Applications to Group Representations*, Kluwer Academic Publishers, 1990.

[13] B. Dubrovin, "Integrable systems in topological field theory," Nucl. Phys. **B379** (1992) 627–689.

[14] T. Nakayama, "On Frobeniusean algebras I," Annals of Math. **40** (1939) 611–633.

[15] S. Katz, "Rational curves on Calabi–Yau threefolds," (revised version), alg-geom/9312009.

[16] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Holomorphic anomalies in topological field theories," with an appendix by S. Katz, Nucl. Phys. **B405** (1993) 298–304, hep-th/9302103.

[17] S. Katz, "Rational curves on Calabi–Yau manifolds: verifying predictions of mirror symmetry," in *Projective Geometry with Applications*, (E. Ballico, editor), Marcel Dekker, 1994, pp. 231–239, alg-geom/9301006.

[18] Y. Ruan, "Topological sigma model and Donaldson type invariants in Gromov theory," preprint, 1993.

[19] Y. Ruan and G. Tian, "A mathematical theory of quantum cohomology," Math. Res. Lett. **1** (1994) 269–278.

[20] D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, American Mathematical Society, 1994.

[21] M. Kontsevich and Yu. Manin, "Gromov–Witten classes, quantum cohomology, and enumerative geometry," Commun. Math. Phys. **164** (1994) 525–562, hep-th/9402147.

[22] M. Kontsevich, "Enumeration of rational curves via torus actions," hep-th/9405035.

[23] Yu. I. Manin, "Generating functions in algebraic geometry and sums over trees," alg-geom/9407005.

[24] B. R. Greene and M. R. Plesser, "Duality in Calabi–Yau moduli space," Nucl. Phys. **B338** (1990) 15–37.

[25] P. Berglund and T. Hübsch, "A generalized construction of mirror manifolds," in *Essays on Mirror Manifolds*, S.-T. Yau, ed. International Press, Hong Kong, 1992, pp. 388–407, hep-th/9201014.

[26] V. V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties," J. Alg. Geom. **3** (1994) 493–535, alg-geom/9310003.

[27] L. A. Borisov, "Towards the mirror symmetry for Calabi–Yau complete intersections in Gorenstein toric Fano varieties," alg-geom/9310001.

[28] V. V. Batyrev and L. A. Borisov, "Dual cones and mirror symmetry for generalized Calabi–Yau manifolds," alg-geom/9402002.

[29] P. Berglund and S. Katz, "Mirror symmetry constructions: a review," preprint IASSNS-HEP-94/38, OSU-M-94/2, to appear in *Essays in mirror manifolds II*, B. R. Greene and S.-T. Yau, eds. International Press, Hong Kong, hep-th/9406008.

[30] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, "Calabi–Yau moduli space, mirror manifolds and spacetime topology change in string theory," Nucl. Phys. **B416** (1994) 414–480, hep-th/9309097.

[31] B. de Wit and A. Van Proeyen, "Potentials and symmetries of general gauged N=2 supergravity-Yang–Mills models," Nucl. Phys. **B245** (1985) 89–117.

[32] B. de Wit, P. Lauwers, and A. Van Proeyen, "Lagrangians of N=2 supergravity-matter systems," Nucl. Phys. **B255** (1985) 569–608.

[33] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, B. de Wit, and L. Girardello, "Vector multiplets coupled to N=2 supergravity: super Higgs effect, flat potentials and geometric structure," Nucl. Phys. **B250** (1985) 385–426.

[34] A. Strominger, "Special geometry," Commun. Math. Phys. **133** (1990) 163–180.

[35] S. Cecotti and C. Vafa, "Topological–anti-topological fusion," Nucl. Phys. **B367** (1991) 359–461.

[36] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165** (1994) 311–427, hep-th/9309140.

[37] B. R. Greene, D. R. Morrison, and M. R. Plesser, "Mirror symmetry in higher dimension," preprint CLNS-93/1253, IASSNS-HEP-94/2, YCTP-P31-92, hep-th/9402119.

[38] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, "A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory," Nucl. Phys. **B359** (1991) 21–74.

[39] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, "The monomial-divisor mirror map," Int. Math. Res. Notices (1993) 319–337, alg-geom/9309007.

[40] D. Mumford, *Geometric Invariant Theory*, Ergeb. Math. Grenzgeb. (N.F.), vol. 34, Springer-Verlag, Berlin, Heidelberg, New York, 1965.

[41] F. C. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Mathematical Notes, no. 31, Princeton University Press, 1984.

[42] L. Ness, "A stratification of the null cone via the moment map," Amer. J. Math. **106** (1984) 1281–1329.

[43] W. Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies, vol. 131, Princeton University Press, 1993.

[44] D. A. Cox, "The homogeneous coordinate ring of a toric variety," Amherst preprint, 1992, alg-geom/9210008.

[45] P. Candelas, X. de la Ossa, A. Font, S. Katz, and D. R. Morrison, "Mirror symmetry for two parameter models (I)," Nucl. Phys. **B416** (1994) 481–538, hep-th/9308083.

[46] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, "Mirror symmetry, mirror map and applications to Calabi–Yau hypersurfaces," Harvard preprint HUTMP-93-0801, hep-th/9308122.

[47] V. V. Batyrev, "On the classification of smooth projective toric varieties," Tôhoku Math. J. **43** (1991) 569–585.

[48] V. Guillemin and S. Sternberg, "Birational equivalence in the symplectic category," Invent. Math. **97** (1989) 485–522.

[49] J. J. Duistermaat and G. J. Heckman, "On the variation in the cohomology of the symplectic form of the reduced phase space," Invent. Math. **69** (1982) 259–269; ibid. **72** (1983) 153–158.

[50] T. Oda and H. S. Park, "Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decompositions," Tôhoku Math. J. **43** (1991) 375–399.

[51] N. Seiberg, "The power of holomorphy: exact results in 4D SUSY field theories," preprint RU-94-64, IASSNS-HEP-94/67, hep-th/9408013, to appear in proceedings of PASCOS 94.

[52] S. Coleman, "More on the massive Schwinger model," Ann. Phys. **101** (1976) 239–267.

[53] S. Cecotti and C. Vafa, “Exact results for supersymmetric sigma models,” Phys. Rev. Lett. **68** (1992) 903–906, hep-th/9111016.

[54] K. Intriligator, R. Leigh, and N. Seiberg, “Exact superpotentials in four dimensions,” Phys. Rev. **D50** (1994) 1092–1104, hep-th/9403198.

[55] S. Cordes, G. Moore, and S. Ramgoolam, “Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories,” preprint YCTP-P11-94, hep-th/9411210.

[56] S. B. Bradlow and G. D. Daskalopoulos, “Moduli of stable pairs for holomorphic bundles over Riemann surfaces,” Int. J. Math. **2** (1991) 477–513.

[57] O. García-Prado, “Dimensional reduction of stable bundles, vortices and stable pairs,” Int. J. Math. **5** (1994) 1–52.

[58] D. A. Cox, “The functor of a smooth toric variety,” Amherst preprint, 1993, alg-geom/9312001.

[59] E. Witten, “The Verlinde algebra and the cohomology of the Grassmannian,” preprint IASSNS-HEP-93/41, hep-th/9312004.

[60] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, “Measuring small distances in N=2 sigma models,” Nucl. Phys. **B420** (1994) 184–242, hep-th/9311042.

[61] B. Siebert and G. Tian, “On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator,” alg-geom/9403010.

[62] E. Witten, “On the structure of the topological phase of two-dimensional gravity,” Nucl. Phys. **B340** (1990) 281–332.

[63] P. S. Aspinwall, “Resolution of orbifold singularities in string theory,” to appear in *Essays on mirror manifolds II*, B. R. Greene and S.-T. Yau, eds. International Press, Hong Kong, hep-th/9403123.

[64] E. Silverstein and E. Witten, “Global $U(1)$ R-symmetry and conformal invariance of $(0,2)$ models,” Phys. Lett. **328B** (1994) 307–311, hep-th/9403054.

[65] P. Berglund and S. Katz, “Mirror symmetry for hypersurfaces in weighted projective space and topological couplings,” Nucl. Phys. **B420** (1994) 289–314, hep-th/9311014.

[66] V. V. Batyrev, “Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori,” Duke Math. J. **69** (1993) 349–409.

[67] V. V. Batyrev and D. A. Cox, “On the Hodge structure of projective hypersurfaces in toric varieties,” Duke Math. J. **75** (1994) 293–338, alg-geom/9306011.

[68] D. A. Cox, “Toric residues,” Amherst preprint, 1994, alg-geom/9410017.

[69] I. M. Gel’fand, A. V. Zelevinskiĭ, and M. M. Kapranov, “Discriminants of polynomials in several variables and triangulations of Newton polyhedra,” Leningrad Math. J. **2** (1991) 449–505 (from Russian in Algebra i analiz **2**).

[70] J. Horn, “Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen,” Math. Ann. **34** (1889) 544–600.

[71] M. M. Kapranov, “A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map,” Math. Ann. **290** (1991) 277–285.

[72] T. Hübsch and S.-T. Yau, “An $SL(2,\mathbb{C})$ action on certain Jacobian rings and the mirror map,” in *Essays on Mirror Manifolds*, S.-T. Yau, ed. International Press, Hong Kong, 1992, pp. 372–387.

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